ECE 534 RANDOM PROCESSES **SPRING 2005**
PROBLEM SET 4 **Due Wednesday, October 19**

Random Processes, including Poisson, Wiener, Markov and Martingale Processes

Assigned Reading: Chapter 4 of the notes.

Reminder: Exam 1, covering lectures, reading, and homework for problem sets 2 and 3 will be held on Monday, October 10, 7-8:15 p.m., in Room 100 MSEB (the second building east of Everitt Lab.) You may bring a sheet of notes, two-sided, font size 10 or larger or equivalent handwriting size, to consult during the exam. Otherwise the exam is closed notes. There will also be classes as usual, including class on Monday morning, October 10.

Problems to be handed in:

1. A simple discrete-time random process
Let $U = (U_n : n \in \mathbb{Z})$ consist of independent random variables, each uniformly distributed on the interval $[0, 1]$. Let $X = (X_k : k \in \mathbb{Z}\}$ be defined by $X_k = \max\{U_{k-1}, U_k\}$.
(a) Sketch a typical sample path of the process X.
(b) Is X stationary?
(c) Is X Markov?
(d) Describe the first order distributions of X.
(e) Describe the second order distributions of X.

2. Invariance of properties under transformations
Let $X = (X_n : n \in \mathbb{Z})$, $Y = (Y_n : n \in \mathbb{Z})$, and $Z = (Z_n : n \in \mathbb{Z})$ be random processes such that $Y_n = X_n^2$ for all n and $Z_n = X_n^3$ for all n. Determine whether each of the following statements is always true. If true, give a justification. If not, give a simple counter example.
(a) If X is Markov then Y is Markov.
(b) If X is Markov then Z is Markov.
(c) If Y is Markov then X is Markov.
(d) If X is stationary then Y is stationary.
(e) If Y is stationary then X is stationary.
(f) If X is wide sense stationary then Y is wide sense stationary.
(g) If X has independent increments then Y has independent increments.
(h) If X is a martingale then Z is a martingale.

3. On an $M/D/\infty$ queue

Suppose packets enter a buffer according to a Poisson point process on $\mathbb{R}$ at rate λ, meaning that the number of arrivals in an interval of length τ has the Poisson distribution with mean $\lambda\tau$, and the numbers of arrivals in disjoint intervals are independent. Suppose each packet stays in the buffer for one unit of time, independently of other packets. Because the arrival process is memoryless, because the service times are deterministic, and because the packets are served simultaneously, corresponding to infinitely many servers, this queueing system is called an $M/D/\infty$ queueing system. The number of packets in the system at time t is given by $X_t = N(t-1,t]$, where $N(a,b]$ is the number of customers that arrive during the interval $(a,b]$.
(a) Find the mean and autocovariance function of X.
(b) Is X stationary? Is X wide sense stationary?
(c) Is X a Markov process?
(d) Find a simple expression for $P\{X_t = 0 \text{ for } t \in [0,1]\}$ in terms of λ.
(e) Find a simple expression for $P\{X_t > 0 \text{ for } t \in [0,1]\}$ in terms of λ.

4. A fly on a cube

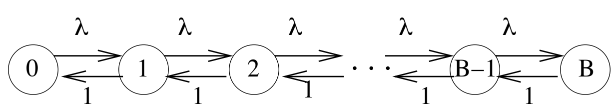
Consider a cube with vertices 000, 001, 010, 100, 110, 101. 011, 111. Suppose a fly walks along edges of the cube from vertex to vertex, and for any integer $t \geq 0$, let X_t denote which vertex the fly is at at time t. Assume $X = (X_t : t \geq 0)$ is a discrete time Markov process, such that given X_t, the next state X_{t+1} is equally likely to be any one of the three vertices neighboring X_t.
(a) Sketch the one step transition probability diagram for X.
(b) Let Y_t denote the distance of X_t, measured in number of hops, between vertex 000 and X_t. For example, if $X_t = 101$, then $Y_t = 2$. The process Y is a Markov process with states 0,1,2, and 3. Sketch the one-step transition probability diagram for Y.
(c) Suppose the fly begins at vertex 000 at time zero. Let τ be the first time that X returns to vertex 000 after time 0, or eqivalently, the first time that Y returns to 0 after time 0. Find $E[\tau]$.

5. A space-time transformation of Brownian motion

Suppose $X = (X_t : t \geq 0)$ is a real-valued, mean zero, independent increment process, and let $E[X_t^2] = \rho_t$ for $t \geq 0$. Assume $\rho_t < \infty$ for all t.
(a) Show that ρ must be nonnegative and nondecreasing over $[0,\infty)$.
(b) Express the autocorrelation function $R_X(s,t)$ in terms of the function ρ for all $s \geq 0$ and $t \geq 0$.
(c) Conversely, suppose a nonnegative, nondecreasing function ρ on $[0,\infty)$ is given. Let $Y_t = W(\rho_t)$ for $t \geq 0$, where W is a standard Brownian motion with $R_W(s,t) = \min\{s,t\}$. Explain why Y is an independent increment process with $E[Y_t^2] = \rho_t$ for all $t \geq 0$.
(d) Define a process Z in terms of a standard Brownian motion W by $Z_0 = 0$ and $Z_t = tW(\frac{1}{t})$ for $t > 0$. Does Z have independent increments? Justify your answer.

6. An M/M/1/B queueing system

Suppose X is a continuous-time Markov process with the transition rate diagram shown, for a positive integer B and positive constant λ.



(a) Find the generator matrix, Q, of X for $B = 5$.

(b) Find the equilibrium probability distribution. (Note: The process X models the number of customers in a queueing system with a Poisson arrival process, exponential service times, one server, and a finite buffer.)

7. A branching process

Let $p = (p_i : i \geq 0)$ be a probability distribution on the nonnegative integers with mean m. Consider a population beginning with a single individual, comprising generation zero. The offspring of the initial individual comprise the first generation, and, in general, the offspring of the kth generation comprise the $k+1^{st}$ generation. Suppose the number of offspring of any individual has the probability distribution p, independently of how many offspring other individuals have. Let $Y_0 = 1$, and for $k \geq 1$ let Y_k denote the number of individuals in the k^{th} generation.

(a) Is $Y = (Y_k : k \geq 0)$ a Markov process? Briefly explain your answer.

(b) Find constants c_k so that $\frac{Y_k}{c_k}$ is a martingale.

(c) Let $a_m = P[Y_m = 0]$, the probability of extinction by the m^{th} generation. Express a_{m+1} in terms of the distribution p and a_m (Hint: condition on the value of Y_1, and note that the Y_1 subpopulations beginning with the Y_1 individuals in generation one are independent and statistically identical to the whole population.)

(d) Express the probability of eventual extinction, $a_\infty = \lim_{m\to\infty} a_m$, in terms of the distribution p. Under what condition is $a_\infty = 1$?

(e) Find a_∞ in terms of θ in case $p_k = \theta^k(1-\theta)$ for $k \geq 0$ and $0 \leq \theta < 1$. (This distribution is similar to the geometric distribution, and it has mean $m = \frac{\theta}{1-\theta}$.)

8. Some orthogonal martingales based on Brownian motion

(This problem is related to the problem on linear innovations and orthogonal polynomials in the previous problem set.) Let $W = (W_t : t \geq 0)$ be a Brownian motion with $\sigma^2 = 1$ (called a standard Brownian motion), and let $M_t = \exp(\theta W_t - \frac{\theta^2 t}{2})$ for an arbitrary constant θ.

(a) Show that $(M_t : t \geq 0)$ is a martingale. (Hint for parts (a) and (b): For notational brevity, let $\mathcal{W}_s$ represent $(W_u : 0 \leq u \leq s)$ for the purposes of conditioning. If Z_t is a function of W_t for each t, then a sufficient condition for Z to be a martingale is that $E[Z_t|\mathcal{W}_s] = Z_s$ whenever $0 < s < t$, because then $E[Z_t|Z_u, 0 \leq u \leq s] = E[E[Z_t|\mathcal{W}_s]|Z_u, 0 \leq u \leq s] = E[Z_s|Z_u, 0 \leq u \leq s] = Z_s$).

(b) By the power series expansion of the exponential function,

$$\begin{aligned}\exp(\theta W_t - \frac{\theta^2 t}{2}) &= 1 + \theta W_t + \frac{\theta^2}{2}(W_t^2 - t) + \frac{\theta^3}{3!}(W_t^3 - 3tW_t) + \cdots \\ &= \sum_{n=0}^{\infty} \frac{\theta^n}{n!} M_n(t)\end{aligned}$$

where $M_n(t) = t^{n/2} H_n(\frac{W_t}{\sqrt{t}})$, and H_n is the n^{th} Hermite polynomial. The fact that M is a martingale for any value of θ can be used to show that M_n is a martingale for each n (you don't need to supply details). Verify directly that $W_t^2 - t$ and $W_t^3 - 3tW_t$ are martingales.

(c) For fixed t, $(M_n(t) : n \geq 0)$ is a sequence of orthogonal random variables, because it is the linear innovations sequence for the variables $1, W_t, W_t^2, \ldots$. Use this fact and the martingale property of the M_n processes to show that if $n \neq m$ and $s, t \geq 0$, then $M_n(s) \perp M_m(t)$.