ECE 534 RANDOM PROCESSES **FALL 2005**
PROBLEM SET 3 **Due Wednesday, October 5**

Random Vectors and Minimum Mean Squared Error Estimation

Assigned Reading: Chapter 3 and the section on matrices in the Appendix, in the course notes.

Problems to be handed in:

1. Calculation of some minimum mean square error estimators
Let $Y = X + N$, where X has the exponential distribution with parameter λ, and N is Gaussian with mean 0 and variance σ^2. The variables X and N are independent, and the parameters λ and σ^2 are strictly positive. (Recall that $E[X] = \frac{1}{\lambda}$ and $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.)
(a) Find $\widehat{E}[X|Y]$ and also find the mean square error for estimating X by $\widehat{E}[X|Y]$.
(b) Does $E[X|Y] = \widehat{E}[X|Y]$? Justify your answer. (Hint: Answer is yes if and only if there is no estimator for X of the form $g(Y)$ with a smaller MSE than $\widehat{E}[X|Y]$.)

2. Conditional probabilities with joint Gaussians I
Let $\begin{pmatrix} X \\ Y \end{pmatrix}$ be a mean zero Gaussian vector with correlation matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, where $|\rho| < 1$.
(a) Express $P[X \leq 1|Y]$ in terms of ρ, Y, and the standard normal CDF, Φ.
(b) Find $E[(X-Y)^2|Y = y]$ for real values of y.

3. Conditional probabilities with joint Gaussians II
Let X, Y be jointly Gaussian random variables with mean zero and covariance matrix

$$\mathrm{Cov}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 18 \end{pmatrix}.$$

You may express your answers in terms of the Φ function defined by $\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds$.
(a) Find $P[|X - 1| \geq 2]$.
(b) What is the conditional density of X given that $Y = 3$? You can either write out the density in full, or describe it as a well known density with specified parameter values.
(c) Find $P[|X - E[X|Y]| \geq 1]$.

4. Some identities for estimators
Let X and Y be random variables with $E[X^2] < \infty$. For each of the following statements, determine if the statement is true. If yes, give a justification using the orthogonality principle. If no, give a counter example.
(a) $E[X\cos(Y)|Y] = E[X|Y]\cos(Y)$
(b) $E[X|Y] = E[X|Y^3]$
(c) $E[X^3|Y] = E[X|Y]^3$
(d) $E[X|Y] = E[X|Y^2]$
(e) $\widehat{E}[X|Y] = \widehat{E}[X|Y^3]$

5. The square root of a positive-semidefinite matrix
(a) True or false? It B is a square matrix over the reals, then BB^T is positive semidefinite.
(b) True or false? If K is a symmetric positive semidefinite matrix over the reals, then there exists a symmetric positive semidefinite matrix S over the reals such that $K = S^2$. (Hint: What if K is also diagonal?)

6. An innovations problem

Let $U_1, U_2, \ldots$ be a sequence of independent random variables, each uniformly distributed on the interval $[0,1]$. Let $Y_0 = 1$, and $Y_n = U_1 U_2 \cdots U_n$ for $n \geq 1$.

(a) Find the variance of Y_n for each $n \geq 1$.

(b) Find $E[Y_n | Y_0, \ldots, Y_{n-1}]$ for $n \geq 1$.

(c) Find $\widehat{E}[Y_n | Y_0, \ldots, Y_{n-1}]$ for $n \geq 1$.

(d) Find the linear innovations sequence $\widetilde{Y} = (\widetilde{Y}_0, \widetilde{Y}_1, \ldots)$.

(e) Fix a positive integer M and let $X_M = U_1 + \ldots + U_M$. Using the answer to part (d), find $\widehat{E}[X_M | Y_0, \ldots, Y_M]$, the best linear estimator of X_M given $(Y_0, \ldots, Y_M)$.

7. Linear Innovations and Orthogonal Polynomials

(a) Let X be a $N(0,1)$ random variable. Show that for integers $n \geq 0$,

$$E[X^n] = \begin{cases} \frac{n!}{(n/2)! 2^{n/2}} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Hint: One approach is to apply the power series expansion for e^x on each side of the identity $E[e^{uX}] = e^{u^2/2}$, and identify the coefficients of u^n.

(b) Let X be a $N(0,1)$ random variable, and let $Y_n = X^n$ for integers $n \geq 0$. Note that $Y_0 \equiv 1$. Express the first five terms of the linear innovations sequence $\widetilde{Y}_n$ in terms of X.

8. Steady state gains for one-dimensional Kalman filter

This is a continuation of Problem 3.7 in the notes, regarding the Kalman filter for a one-dimensional state space.

(a) Show that $\lim_{k \to \infty} \sigma_k^2$ exists.

(b) Express the limit, σ_∞^2, in terms of f.

(c) Explain why $\sigma_\infty^2 = 1$ if $f = 0$.