ECE 534 RANDOM PROCESSES **FALL 2005**
PROBLEM SET 2 **Due September 21**

Sequences of Random Variables

Assigned Reading: Chapter 2 and Sections 8.1-8.3 of the course notes. Additional material on limits for deterministic sequences can be found in Kenneth Ross, *Elementary Analysis: The Theory of Calculus*, Sections 7-10 (pp. 24-48). This book is on reserve in the mathematics library in Altgeld Hall, and it is required for Math 447: *Introduction to Higher Analysis: Real Variables.*

Reminders: A one hour quiz on probability will be given 7-8 p.m. Monday, September 12, Room to be announced.

Problems to be handed in:

1. On convergence of deterministic sequences and functions
(a) Let $x_n = \frac{8n^2+n}{3n^2}$ for $n \geq 1$. Prove that $\lim_{n\to\infty} x_n = \frac{8}{3}$.
(b) Suppose f_n is a function on some set D for each $n \geq 1$ and f, and suppose f is also a function on D. Then f_n is defined to converge to f *uniformly* if for any $\epsilon > 0$, there exists an n_ϵ such that $|f_n(x) - f(x)| \leq \epsilon$ for all $x \in D$ whenever $n \geq n_\epsilon$. A key point is that n_ϵ does not depend on x. Show that the functions $f_n(x) = x^n$ on the semi-open interval $[0, 1)$ do not converge uniformly to the zero function.
(c) The supremum of a function f on D, written $\sup_D f$, is the least upper bound of f. Equivalently, $\sup_D f$ satisfies $\sup_D f \geq f(x)$ for all $x \in D$, and given any $c < \sup_D f$, there is an $x \in D$ such that $f(x) \geq c$. Show that $|\sup_D f - \sup_D g| \leq \sup_D |f - g|$. Conclude that if f_n converges to f uniformly on D, then $\sup_D f_n$ converges to $\sup_D f$.

2. Convergence of sequences of random variables
Let Θ be uniformly distributed on the interval $[0, 2\pi]$. In which of the four senses (*a.s., m.s., p., d.*) do each of the following two sequences converge. Identify the limits, if they exist, and justify your answers.
(a) $(X_n : n \geq 1)$ defined by $X_n = \cos(n\Theta)$.
(b) $(Y_n : n \geq 1)$ defined by $Y_n = |1 - \frac{\Theta}{\pi}|^n$.

3. Convergence of a minimum
Let $U_1, U_2, \ldots$ be a sequence of independent random variables, with each variable being uniformly distributed over the interval $[0, 1]$, and let $X_n = \min\{U_1, \ldots, U_n\}$ for $n \geq 1$.
(a) Determine in which of the senses (a.s., m.s., p., d.) the sequence (X_n) converges as $n \to \infty$, and identify the limit, if any. Justify your answers.
(b) Determine the value of the constant θ so that the sequence (Y_n) defined by $Y_n = n^\theta X_n$ converges in distribution as $n \to \infty$ to a nonzero limit, and identify the limit distribution.

4. Convergence of a product
Let $U_1, U_2, \ldots$ be a sequence of independent random variables, with each variable being uniformly distributed over the interval $[0, 2]$, and let $X_n = U_1 U_2 \cdots U_n$ for $n \geq 1$.
(a) Determine in which of the senses (a.s., m.s., p., d.) the sequence (X_n) converges as $n \to \infty$, and identify the limit, if any. Justify your answers.
(b) Determine the value of the constant θ so that the sequence (Y_n) defined by $Y_n = n^\theta \log(X_n)$ converges in distribution as $n \to \infty$ to a nonzero limit.

5. Limit behavior of a stochastic dynamical system

Let $W_1, W_2, \ldots$ be a sequence of independent, $N(0, 0.5)$ random variables. Let $X_0 = 0$, and define $X_1, X_2, \ldots$ recursively by $X_{k+1} = X_k^2 + W_k$. Determine in which of the senses (a.s., m.s., p., d.) the sequence (X_n) converges as $n \to \infty$, and identify the limit, if any. Justify your answer.

6. Applications of Jensen's inequality

Explain how each of the inequalties below follows from Jensen's inequality. Specifically, identify the convex function and random variable used.

(a) $E[\frac{1}{X}] \geq \frac{1}{E[X]}$, for a positive random variable X with finite mean.

(b) $E[X^4] \geq E[X^2]^2$, for a random variable X with finite second moment.

(c) $D(f|g) \geq 0$, where f and g are positive probability densities on a set A, and D is the divergence distance defined by $D(f|g) = \int_A f(x) \log \frac{f(x)}{g(x)} dx$. (The base used in the logarithm is not relevant.)

7. Chernoff bound for Gaussian and Poisson random variables

(a) Let X have the $N(\mu, \sigma^2)$ distribution. Find the optimized Chernoff bound on $P\{X \geq E[X]+c\}$ for $c \geq 0$.

(b) Let Y have the $Poi(\lambda)$ distribution. Find the optimized Chernoff bound on $P\{Y \geq E[Y] + c\}$ for $c \geq 0$.

(c) (The purpose of this problem is to highlight the similarity of the answers to parts (a) and (b).) Show that your answer to part (b) can be expressed as $P\{Y \geq E[Y]+c\} \leq \exp(-\frac{c^2}{2\lambda}\psi(\frac{c}{\lambda}))$ for $c \geq 0$, where $\psi(u) = 2g(1+u)/u^2$, with $g(s) = s(\log s - 1) + 1$. (Note: Y has variance λ, so the essential difference between the normal and Poisson bounds is the ψ term. The function ψ is strictly positive and strictly decreasing on the interval $[-1, +\infty)$, with $\psi(-1) = 2$ and $\psi(0) = 1$. Also, $u\psi(u)$ is strictly increasing in u over the interval $[-1, +\infty)$.)

8. Large deviations of a mixed sum

Let $X_1, X_2, \ldots$ have the $Exp(1)$ distribution, and $Y_1, Y_2, \ldots$ have the $Poi(1)$ distribution. Suppose all these random variables are mutually independent. Let $0 \leq f \leq 1$, and suppose $S_n = X_1 + \cdots + X_{nf} + Y_1 + \cdots + Y_{(1-f)n}$. Define $l(f, a) = \lim_{n\to\infty} \frac{1}{n} \ln P\{\frac{S_n}{n} \geq a\}$ for $a > 1$. Cramérs theorem can be extended to show that $l(f, a)$ can be computed by replacing the probability $P\{\frac{S_n}{n} \geq a\}$ by its optimized Chernoff bound. (For example, if $f = 1/2$, we simply view S_n as the sum of the $\frac{n}{2}$ i.i.d. random variables, $X_1 + Y_1, \ldots, X_{\frac{n}{2}} + Y_{\frac{n}{2}}$.) Compute $l(f, a)$ for $f \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ and $a = 4$.