ECE 534 RANDOM PROCESSES **FALL 2005**
PROBLEM SET 1 **Due September 7**

Please visit the course website soon: `courses.ece.uiuc.edu/ece534/`

1. Review of Basic Probability

Assigned reading: Chapter one, "Getting started," of the course notes, including the problems at the end of the chapter. It would also be good to review some of the final exams from the last six semesters of ECE413. The past ECE413 homeworks, exams, and their solutions are available on the ECE 413 website.

Problems to be handed in:

1. Conditional probabilities–basic computations of iterative decoding
(a) Suppose $B_1, \ldots, B_n, Y_1, \ldots, Y_n$ are discrete random variables with joint pmf

$$p(b_1, \ldots, b_n, y_1, \ldots, y_n) = \begin{cases} 2^{-n} \prod_{i=1}^n q_i(y_i|b_i) & \text{if } b_i \in \{0,1\} \text{ for } 1 \leq i \leq n \\ 0 & \text{else} \end{cases}$$

where $q_i(y_i|b_i)$ as a function of y_i is a pmf for $b_i \in \{0,1\}$. Finally, let $B = B_1 \oplus \cdots \oplus B_n$ represent the modulo two sum of $B_1, \ldots, B_n$. Thus, the ordinary sum of the $n+1$ binary random variables $B_1, \ldots, B_n, B$ is even. Express $P[B = 1|Y_1 = y_1, \ldots, Y_n = y_n]$ in terms of the y_i and the functions q_i. Simplify your answer.
(b) Suppose B and $Z_1, \ldots, Z_k$ are discrete random variables with joint pmf

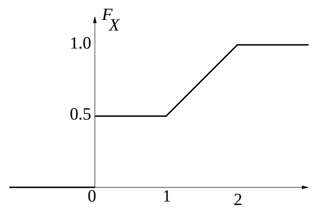
$$p(b, z_1, \ldots, z_k) = \begin{cases} \frac{1}{2} \prod_{j=1}^n r_j(z_j|b) & \text{if } b \in \{0,1\} \\ 0 & \text{else} \end{cases}$$

where $r_j(z_j|b)$ as a function of z_j is a pmf for $b \in \{0,1\}$ fixed. Express $P[B = 1|Z_1 = z_1, \ldots, Z_k = z_k]$ in terms of the z_j and the functions r_j.

2. Blue corners
Suppose each corner of a cube is colored blue, independently of the other corners, with some probability p. Let B denote the event that at least one face of the cube has all four corners colored blue. (a) Find the conditional probability of B given that exactly five corners of the cube are colored blue. (b) Find $P(B)$, the unconditional probability of B.

3. Congestion at output ports
Consider a packet switch with some number of input ports and eight output ports. Suppose four packets simultaneously arrive on different input ports, and each is routed toward an output port. Assume the choices of output ports are mutually independent, and for each packet, each output port has equal probability.
(a) Describe a probability space $(\Omega, \mathcal{F}, P)$ to describe this situation.
(b) Let X_i denote the number of packets routed to output port i, for $1 \leq i \leq 8$. Describe the joint pmf of $X_1, \ldots, X_8$.
(c) Find $\mathrm{Cov}(X_1, X_2)$.
(d) Find $P\{X_i \leq 1 \text{ for all } i\}$
(e) Find $P\{X_i \leq 2 \text{ for all } i\}$

4. A CDF of mixed type
Let X have the CDF shown.



(a) Find $P\{X \leq 0.8\}$.
(b) Find $E[X]$
(c) Find $Var(X)$

5. Density of a function of a random variable
Suppose X is a random variable with probability density function

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

(a) Find $P[X \geq 0.4 | X \leq 0.8]$.
(b) Find the density function of Y defined by $Y = -\log(X)$.

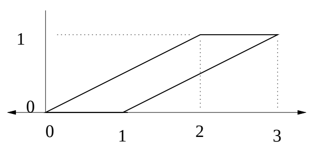
6. Moments and densities of functions of a random variable
Suppose the length L and width W of a rectangle are independent and each uniformly distributed over the interval $[0, 1]$. Let $C = 2L + 2W$ (the length of the perimeter) and $A = LW$ (the area). Find the means, variances, and probability densities of C and A.

7. Gaussians and the Q function
Let X and Y be independent, $N(0, 1)$ random variables.
(a) Find $\text{Cov}(3X + 2Y, X + 5Y + 10)$.
(b) Express $P\{X + 4Y \geq 2\}$ in terms of the Q function defined by $Q(u) = \int_{v=u}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$.
(c) Express $P\{(X - Y)^2 > 9\}$ in terms of the Q function.

8. Density of a difference
Let X and Y be independent, exponentially distributed random variables with parameter λ, such that $\lambda > 0$. Find the probability density of $Z = |X - Y|$.

9. Working with a two dimensional density

Let the random variables X and Y be jointly uniformly distributed over the region shown.



(a) Determine the value of f_{XY} on the region shown.
(b) Find f_X, the marginal pdf of X.
(c) Find the mean and variance of X.
(d) Find the conditional pdf of Y given that $X = x$, for $0 \leq x \leq 1$.
(e) Find the conditional pdf of Y given that $X = x$, for $1 \leq x \leq 2$
(f) Find and sketch $E[Y|X = x]$ as a function of x. Be sure to specify which range of x this conditional expectation is well defined for.

10. Jointly distributed variables

Let U and V be independent random variables, such that U is uniformly distributed over the interval $[0, 1]$, and V has the exponential probability density function with parameter $\lambda > 0$.
(a) Calculate $E[\frac{V^2}{1+U}]$.
(b) Calculate $P\{U \leq V\}$.
(c) Find the joint probability density function of Y and Z, where $Y = U^2$ and $Z = UV$.