University of Illinois at Urbana-Champaign

ECE 534: Random Processes

Fall 2005
Final Exam

Saturday, December 17, 2005

Name: ____

- You have three hours for this exam. The exam is closed book and closed note, except you may consult both sides of three $8.5'' \times 11''$ sheets of notes in ten point font size or larger, or equivalent handwriting size.
- Calculators, laptop computers, Palm Pilots, two-way e-mail pagers, etc. may not be used.
- Write your answers in the spaces provided.
- **Please show all of your work. Answers without appropriate justification will receive very little credit.** If you need extra space, use the back of the previous page.

Score:

1. __________ (12 pts.)

2. __________ (9 pts.)

3. __________ (12 pts.)

4. __________ (6 pts.)

5. __________ (9 pts.)

6. __________ (12 pts.)

Total: __________(60 pts.)

Problem 1 *(12 points)* Let $U = (U_k : k \in \mathbb{Z})$ such that the U_k are independent, and each is uniformly distributed on the interval $[0, 1]$. Let $X = (X_t : t \in \mathbb{R})$ denote the continuous time random process obtained by linearly interpolating between the U's. Specifically, $X_n = U_n$ for any $n \in \mathbb{Z}$, and X_t is affine on each interval of the form $[n, n+1]$ for $n \in \mathbb{Z}$.
(a) Sketch a sample path of U and a corresponding sample path of X.

(b) Let $t \in \mathbb{R}$. Find and sketch the first order marginal density, $f_{X,1}(x,t)$. (Hint: Let $n = \lfloor t \rfloor$ and $a = t - n$, so that $t = n + a$. Then $X_t = (1-a)U_n + aU_{n+1}$. It's helpful to consider the cases and $0.5 < a < 1$ separately. For brevity, you need only consider the case $0 \leq a \leq 0.5$.)

(c) Is the random process X WSS? Justify your answer.

(d) Find $P\{\max_{0 \leq t \leq 10} X_t \leq 0.5\}$.

Problem 2 *(9 points)* Let $X = (X_t : t \in \mathbb{Z})$ be a real stationary Gaussian process with mean zero and $R_X(t) = \frac{1}{1+t^2}$. Answer the following unrelated questions.
(a) Is X a Markov process? Justify your anwer.

(b) Find $E[X_3|X_0]$ and express $P\{|X_3 - E[X_3|X_0]| \geq 10\}$ in terms of Q, the standard Gaussian complementary cumulative distribution function.

(c) Describe the joint probability density of $(X_0, X_1, X_2)^T$. You need not write it down in detail.

Problem 3 *(12 points)* Let $U = (U_k : k \in \mathbb{Z})$ be a random process such that the variables U_k are independent, identically distributed, with $E[U_k] = \mu$ and $\mathrm{Var}(U_k) = \sigma^2$, where $\mu \neq 0$ and $\sigma^2 > 0$. Please keep in mind that $\mu \neq 0$.
Let $X = (X_n : n \in \mathbb{Z})$ be defined by $X_n = \sum_{k=0}^{\infty} U_{n-k} a^k$, for a constant a with $0 < a < 1$.
(a) Is X stationary? Find the mean function μ_X and autocovariance function C_X for X.

(b) Is X mean ergodic in the m.s. sense?

Let U be as before, and let $Y = (Y_n : n \in \mathbb{Z})$ be defined by $Y_n = \sum_{k=0}^{\infty} U_{n-k} A^k$, where A is a random variable distributed on the interval $(0, 0.5)$ (the exact distribution is not specified), and A is independent of the random process U.
(c) Is Y stationary? Find the mean function μ_Y and autocovariance function C_Y for Y.

(d) Is Y mean ergodic in the m.s. sense?

Problem 4 *(6 points)* Consider a time-homogeneous, discrete-time Markov process $X = (X_k : k \geq 0)$ with state space $\mathcal{S} = \{1, 2, 3\}$, initial state $X_0 = 3$, and one-step transition probability matrix

$$P = \begin{pmatrix} 0.0 & 0.8 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.2 & 0.8 & 0.0 \end{pmatrix}.$$

(a) Sketch the transition probability diagram and find the equilibrium probability distribution $\pi = (\pi_1, \pi_2, \pi_3)$.

(b) Identify a function f on $\mathcal{S}$ so that $f(s) = a$ for two choices of s and $f(s) = b$ for the third choice of s, such that the process $Y = (Y_k : k \geq 0)$ defined by $Y_k = f(X_k)$ is a Markov process, and give the one-step transition probability matrix of Y. Briefly explain your answer.

Problem 5 *(9 points)* Let X be a mean zero, WSS random process with power spectral density $S_X(\omega) = \frac{1}{\omega^4+5\omega^2+4}$.

(a) Find the positive type, minimum phase rational function S_X^+ such that $S_X(\omega) = |S_X^+(\omega)|^2$.

(b) Let T be a fixed known constant with $T \geq 0$. Using the formula $H = \frac{1}{S_X^+}\left[S_X^+ e^{j\omega T}\right]_+$, find $\widehat{X}_{t+T|t}$, the MMSE linear estimator of X_{t+T} given $(X_s : s \leq t)$. Be as explicit as possible. (Hint: Convert to the time domain at the end. Check that your answer is correct in case $T = 0$ and in case $T \to \infty$).

(c) Find the MSE for the optimal estimator of part (b).

Problem 6 *(12 points)* (Note: This problem is about solving the same prediction problem solved in the previous problem. The two problems will be graded separately, so your solutions should be independent.) Suppose that W is a Gaussian white noise process with $R_W(\tau) = \delta(\tau)$. Let X be the stationary random process solving the following second order linear stochastic differential equation: $X'' + 3X' + 2X = W$, or equivalently, Z defined by $Z_t = \begin{pmatrix} X_t \\ X_t' \end{pmatrix}$ satisfies $Z' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} Z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} W$.

(a) Show that $S_X(\omega) = \frac{1}{\omega^4+5\omega^2+4}$.

(b) Find $R_X(\tau)$ for $\tau \in \mathbb{R}$. (Hint: Use a partial fraction expansion of S_X.)

(c) The process X is mean square differentiable. Find $R_{X'}(\tau)$ and $R_{XX'}(\tau)$ for $\tau \in \mathbb{R}$. (Hint: Check that $R_{XX'}(0) = 0$.)

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(d) Explain why Z is a Markov process.

(e) Let $T > 0$, let $t \in \mathbb{R}$, and let $\widehat{X}_{t+T|t}$ be the MMSE estimator of X_{t+T} given $(X_u : u \leq t)$, which is the same as the MMSE estimator of X_{t+T} given $(Z_u : u \leq t)$, because $(X_u : u \leq t)$ and $(Z_u : u \leq t)$ are linearly equivalent. Explain why $\widehat{X}_{t+T|t}$ is a linear combination of X_t and X_t'.

(f) Using parts (c) and (e), find $\widehat{X}_{t+T|t}$.