University of Illinois at Urbana-Champaign

ECE 534: Random Processes

Fall 2005
Exam 2

Monday, November 7, 2005

Name: ____

- You have 75 minutes for this exam. The exam is closed book and closed note, except you may consult both sides of two $8.5'' \times 11''$ sheets of notes in ten point font size or larger, or equivalent handwriting size.
- Calculators, laptop computers, Palm Pilots, two-way e-mail pagers, etc. may not be used.
- Write your answers in the spaces provided.
- **Please show all of your work. Answers without appropriate justification will receive very little credit.** If you need extra space, use the back of the previous page.

Score:

1. __________ (9 pts.)

2. __________ (9 pts.)

3. __________ (6 pts.)

Total: ________(24 pts.)

Problem 1 *(9 points)* Let $N = (N_t : t \geq 0)$ be a Poisson process with rate $\lambda > 0$.
(a) Give a simple expression for $P[N_1 \geq 1 | N_2 = 2]$ in terms of λ.

(b) Give a simple expression for $P[N_2 = 2 | N_1 \geq 1]$ in terms of λ.

(c) Let $X_t = N_t^2$. Is $X = (X_t : t \geq 0)$ a time-homogeneous Markov process? If so, give the transition probabilities $p_{ij}(\tau)$. If not, explain.

Problem 2 *(9 points)* Suppose $X = (X_t : a \leq t \leq b)$ is a m.s. continuous process, and that the Mercer series expansion of R_X is given by

$$R_X(s,t) = \sum_{n=1}^{\infty} \lambda_n \phi_n(s)\phi_n^*(t) \qquad s,t \in [a,b]$$

where the functions $\phi_1, \phi_2, \ldots$ form a complete orthonormal basis for $L^2[a,b]$, and $\lambda_1 \geq \lambda_2 \geq \cdots$
(a) Identify a function $h \in L^2[a,b]$ with $||h||^2 = 1$, which maximizes $E[|\int_a^b X_t h^*(t)dt|^2]$.

(b) Suppose f and g are orthogonal functions, i.e. $(f,g) = \int_a^b f(t)g^*(t)dt = 0$. Are the random variables (X,f) and (X,g) necessarily orthogonal? Explain.

(c) Is the following necessarily true? If X is m.s. differentiable, then the Karhunen-Lòeve expansion of X' can be given with eigenfunctions $\psi_n(t) = \phi_n'(t)/||\phi_n'||$ (normalized versions of the derivatives of the ϕ_n) and eigenvalues $\mu_n = \lambda_n ||\phi_n'||^2$. Explain.

Problem 3 *(6 points)* Suppose at time $k = 2$, there is a bag with two balls in it, one orange and one blue. During each time step, one of the balls is selected from the bag at random, with all balls in the bag having equal probability. That ball, and a new ball of the same color, are both put into the bag. Thus, at time k there are k balls in the bag, for all $k \geq 2$. Let X_k denote the number of blue balls in the bag at time k.

(a) Is $X = (X_k : k \geq 2)$ a Markov process?

(b) Let $M_k = \frac{X_k}{k}$. Thus, M_k is the fraction of balls in the bag at time k that are blue. Determine whether $M = (M_k : k \geq 2)$ is a martingale.