University of Illinois at Urbana-Champaign

ECE 534: Random Processes

Fall 2005
Exam I

Monday, October 10, 2005

Name: ____

- You have 75 minutes for this exam. The exam is closed book and closed note, except you may consult both sides of one $8.5'' \times 11''$ sheet of notes in ten point font size or larger, or equivalent handwriting size.
- Calculators, laptop computers, Palm Pilots, two-way e-mail pagers, etc. may not be used.
- Write your answers in the spaces provided.
- **Please show all of your work. Answers without appropriate justification will receive very little credit.** If you need extra space, use the back of the previous page.

Score:

1. __________ (12 pts.)

2. __________ (12 pts.)

3. __________ (6 pts.)

Total: ________(30 pts.)

Problem 1 *(12 points)* Let $A_1, A_2, \ldots$ be a sequence of independent random variables, with $P[A_i = 1] = P[A_i = \frac{1}{2}] = \frac{1}{2}$ for all i. Let $B_k = A_1 \cdots A_k$.

(a) Does $\lim_{k\to\infty} B_k$ exist in the m.s. sense? Justify your anwswer.

(b) Does $\lim_{k\to\infty} B_k$ exist in the a.s. sense? Justify your anwswer.

(c) Let $S_n = B_1 + \ldots + B_n$. You can use without proof (time is short!) the fact that $\lim_{m,n\to\infty} E[S_m S_n] = \frac{35}{3}$, which implies that $\lim_{n\to\infty}\ S_n$ exists in the m.s. sense. Find the mean and variance of the limit random variable.

(d) Does $\lim_{n\to\infty}\ a.s.\ S_n$ exist? Justify your anwswer.

Problem 2 *(12 points)* Let X, Y, and Z be random variables with finite second moments and suppose X is to be estimated. For each of the following, if true, give a brief explanation. If false, give a counter example.

(a) TRUE or FALSE: $E[|X - E[X|Y]|^2] \leq E[|X - \widehat{E}[X|Y, Y^2]|^2]$.

(b) TRUE or FALSE: $E[|X - E[X|Y]|^2] = E[|X - \widehat{E}[X|Y, Y^2]|^2]$ if X and Y are jointly Gaussian.

(c) TRUE or FALSE? $E[\ |X - \ E[E[X|Z]\ |Y]|^2] \leq E[|X - E[X|Y]|^2]$.

(d) TRUE or FALSE? If $E[|X - E[X|Y]|^2] = \mathrm{Var}(X)$, then X and Y are independent.

Problem 3 *(6 points)* Recall from a homework problem that if $0 < f < 1$ and if S_n is the sum of n independent random variables, such that a fraction f of the random variables have a CDF F_Y and a fraction $1-f$ have a CDF F_Z, then the large deviations exponent for $\frac{S_n}{n}$ is given by:

$$l(a) = \max_{\theta} \{\theta a - f M_Y(\theta) - (1-f) M_Z(\theta)\}$$

where $M_Y(\theta)$ and $M_Z(\theta)$ are the log moment generating functions for F_Y and F_Z respectively.

Consider the following variation. Let $X_1, X_2, \ldots, X_n$ be independent, and identically distributed, each with CDF given by $F_X(c) = f F_Y(c) + (1-f) F_Z(c)$. Equivalently, each X_i can be generated by flipping a biased coin with probability of heads equal to f, and generating X_i using CDF F_Y if heads shows and generating X_i with CDF F_Z if tails shows. Let $\widetilde{S}_n = X_1 + \cdots + X_n$, and let $\widetilde{l}$ denote the large deviations exponent for $\frac{\widetilde{S}_n}{n}$.

(a) Express the function $\widetilde{l}$ in terms of f, M_Y, and M_Z.

(b) Determine which is true and give a proof: $\widetilde{l}(a) \leq l(a)$ for all a, or $\widetilde{l}(a) \geq l(a)$ for all a.