Problem 1. Recall that a mapping $u \mapsto y$ is **finite-gain $\mathcal{L}_2$ stable** if there exists constants γ and β such that $\|y\|_{\mathcal{L}_2} \leq \gamma\|u\|_{\mathcal{L}_2} + \beta$. Here, the $\mathcal{L}_2$ norm is: $\|x\|_{\mathcal{L}_2} = \left(\int_0^T |x(t)|^2 dt\right)^{1/2}$.

Consider the feedback interconnection in Figure 1.

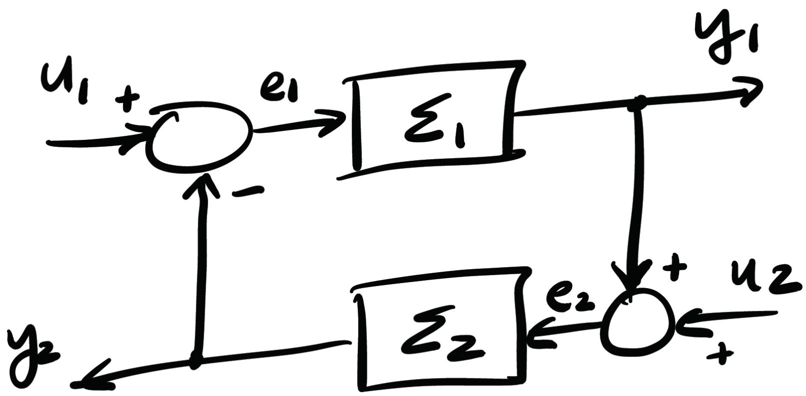

Figure 1: The feedback interconnection of Σ_1 and Σ_2.

Show that the mapping from (u_1, u_2) to (y_1, y_2) is finite-gain $\mathcal{L}_2$ stable if and only if the mapping from (u_1, u_2) to (e_1, e_2) is finite-gain $\mathcal{L}_2$ stable.

Problem 2. Consider the following system:

$$\dot{x} = f(x, u) \qquad y = h(x, u)$$

Recall that this system is **output strictly passive** if there exists a storage function $V(x)$ such that $\dot{V}(x, u) \leq u^\intercal y - y^\intercal \rho(y)$ for some ρ with $y^\intercal \rho(y) > 0$ for any $y \neq 0$.[1] Also, note the definition of finite-gain $\mathcal{L}_2$ stable from the previous problem.

Suppose Σ_1 is output strictly passive with $\rho_1(y_1) = \delta_1 y_1$ and similarly Σ_2 with $\rho_2(y_2) = \delta_2 y_2$.

Show that the feedback interconnection of Figure 1 is finite-gain $\mathcal{L}_2$ stable, with $\gamma \leq 1/\min(\delta_1, \delta_2)$.

[1] Recall also that a storage function is a continuously differentiable positive semidefinite function.

Problem 3. In this problem, we consider the Popov criterion for absolute stability. You will need the general form of the Kalman-Yakubovich-Popov lemma, which can be found on pg. 240 (Lemma 6.3) in Khalil.

We consider the system:

$$\dot{x} = Ax - b\varphi(c^\intercal x) \tag{1}$$

As discussed in lecture, we view this as the interconnection between the following SISO LTI system and static nonlinearity:

$$\dot{x} = Ax + bu \qquad y = c^\intercal x$$
$$u = -\varphi(y)$$

We've already shown that when $g(s)$ is strictly positive real and φ is any passive nonlinearity (i.e. $y\varphi(y) \geq 0$ for all y), then the closed-loop system (Equation (1)) is globally asymptotically stable. (Of course, this is subject to some well-posedness of the feedback loop. Continuity of φ is sufficient to guarantee well-posedness.)

Now, let's generalize this a bit. Suppose:

- For some $\alpha > 0$, $(1 + \alpha s)g(s)$ is strictly positive real and $-1/\alpha$ is not an eigenvalue of A.
- $\varphi(\cdot)$ is continuous and $y\varphi(y) \geq 0$ for all y.

(Note that the case where $g(s)$ is strictly positive real could be thought of as the special case where $\alpha = 0$.)

Show that the closed-loop system is globally asymptotically stable for any such nonlinearity φ.

Hint: Don't forget that the KYP lemma requires minimality of the state-space model in order to be invoked. Consider the Lyapunov function $V(x) = \frac{1}{2}x^\intercal Px + \alpha \int_0^{c^\intercal x} \varphi(z)dz$.