Problem 1. Recall the following definitions.

A sequence $(x_n)_n$ converges to a point x^* if: for any $\epsilon > 0$, there exists an N such that for any $n \geq N$, we have $|x_n - x^*| < \epsilon$. We denote this as $x_n \to x^*$.

Analogously, we can define this for trajectories, e.g. functions of time. A function $x(\cdot)$ with domain $[0, \infty)$ converges to a point x^* if: for any $\epsilon > 0$, there exists a T such that for any $t \geq T$, we have $|x(t) - x^*| < \epsilon$. We also denote this as $x(t) \to x^*$.

In both cases, the idea is the same: for arbitrarily small distances ϵ, after a certain point in time (N or T), we will be within a distance ϵ forever afterward.

We can also define convergence to infinity. $x_n \to +\infty$ means that for any $K > 0$, there exists a N such that $n \geq N$ implies $x_n > K$. These can similarly be extended to trajectories, as well as negative infinity as the limit.

Now, prove the following two statements are equivalent.

1. $x(t) \to x^*$.
2. For any sequence of times $(t_n)_n$, where $t_0 < t_1 < t_2 < \dots$ and $t_n \to +\infty$, we have the sequence $(x(t_n))_n$ converges to x^*.

Hint: If you haven't seen it before, you should be aware of proof by contrapositive. Namely, $p \to q$ is equivalent to $\neg q \to \neg p$. So, to show equivalence $p \leftrightarrow q$, we can show $p \to q$ and $\neg p \to \neg q$.

Problem 2. Consider a trajectory $x(t)$ of the autonomous system $\dot{x}(t) = f(x(t))$, and let x^* be a stable equilibrium. (Here, we mean stable in the sense of Lyapunov.)

Prove the following statement: if there exists a time sequence $(t_n)_n$ where $t_0 < t_1 < t_2 < \dots$ such that $x(t_n) \to x^*$, then $x(t) \to x^*$.

Problem 3. Consider the non-autonomous, 1-D system:

$$\dot{x}(t) = -\frac{2tx(t)}{1+t^2}$$

Is the system stable? Asymptotically stable? Globally asymptotically stable? Globally uniformly asymptotically stable? Justify your answer.

(As per common conventions, all these questions refer to the equilibrium $x^* = 0$.)

Problem 4. Suppose that a function $V : \mathbb{R}^n \to \mathbb{R}$ is positive definite and radially unbounded. Thus, there exists $\mathcal{K}_\infty$ functions α_1 and α_2 such that:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

Suppose the derivative of V along solutions of the system $\dot{x} = f(x)$ satisfies $\dot{V} \leq 0$. It follows from Lyapunov's stability theorem that the system is stable in the sense of Lyapunov: for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|x(0)| \leq \delta$ implies $|x(t)| \leq \epsilon$ for all $t \geq 0$.

Derive a specific expression for δ as a function of ϵ, in terms of the functions α_1 and α_2.

Problem 5. Consider the following system:

$$\ddot{x} + \psi(x)\dot{x} + \varphi(x) = 0$$

Prove the system is globally asymptotically stable for all continuous functions ψ and φ satisfying the following conditions:

1. $x\varphi(x) > 0$ for all $x \neq 0$.
2. $\psi(x) > 0$ for all $x \neq 0$.
3. The function $\Phi(x) = \int_0^x \varphi(z)dz$ is radially unbounded.

Note: You will have to show that the origin is an equilibrium as well, since this is not explicitly given in the problem.

Problem 6. Consider a positive definite, continuously differentiable (C^1) function V that satisfies the following property: there exists an $\epsilon > 0$ such that $\dot{V}(x) > 0$ for all x in the set $\{x : 0 < |x| \leq \epsilon\}$. Here, $\dot{V}(x)$ denotes the derivative of V along solutions of the autonomous system $\dot{x} = f(x)$.

Prove that the system is not stable.

Note: You are expected to show this directly. It is worth noting that this is a consequence of Chetaev's instability theorem (Khalil, Theorem 4.3). However, you may not invoke this theorem in your solution. Additionally, you are encouraged to attempt this proof prior to reading the proof of Theorem 4.3.