Problem 1. Let $D \subseteq \mathbb{R}^n$ be a nonempty, closed set, and let $f : D \to \mathbb{R}^n$ be a continuous function. Show that the fixed points of f, i.e. the set $\{x \in D : f(x) = x\}$, is closed.

Problem 2. Consider the recursive equation:

$$x_{k+1} = \frac{x_k}{2} + \frac{2}{x_k}$$

Show that this sequence converges to 2 for any initial condition $x_0 \in [\sqrt{2}, 2\sqrt{2}]$.

Problem 3. Suppose the function $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x uniformly across t in $[t_0, t_1]$. Let W be a compact subset of $\mathbb{R}^n$. Show that $\sup_{t \in [t_0,t_1], x \in W} |f(t, x)|$ is finite.

Hint: One common way to show a function is bounded everywhere is to fix any point (s, y) in the domain. Noting that $f(s, y)$ is finite, we can show the function itself is bounded if we can bound how far away $f(t, x)$ is from $f(s, y)$ for any other (t, x). To start, it may be easier to consider the case where $f(t, x)$ is continuous in t. Then, note that piecewise continuity states that the interval can be broken up into finitely many intervals where it is continuous, and the endpoints of the intervals agree with either the right or left limit.

Problem 4. Consider:

$$\dot{x}(t) = f(t, x(t))$$

Suppose the function $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x uniformly across t in $[t_0, t_1]$. Let W be a compact subset of $\mathbb{R}^n$. Prove that there exists a $\delta > 0$ such that every solution such that $x(t_0) \in W$ can be extended to be defined on $[t_0, t_0 + \delta]$.

Note: Here, δ depends on the compact set W, but not on the initial condition $x(t_0)$. Thus, if it is known that $x(t)$ remains in W, then we can define a solution globally, without assuming global Lipschitzness on f. We will see later in this course other ways to prove this boundedness of trajectories.

Hint: This problem can be done with the appropriate modifications to the existence and uniqueness proof covered in class. Recall that compactness in $\mathbb{R}^n$ means closed and bounded. In this setting, we can always 'grow our set a little bit' and still be closed and bounded. So, for trajectories that start in W, we can grow W to a slightly larger, still bounded set U, and see that our trajectories will stay in U for a small period of time.