ECE 515/ME 540: Problem Set 12: Problems and Solutions
Introduction to the Minimum Principle

Due: Wednesday, December 11, 11:59pm
Reading: Course notes, Chapter 11

1. [**Lagrange multipliers – sensitivity interpretation**]
Suppose ϕ_i for $1 \leq i \leq K$ are continuously differentiable strictly convex functions on the reals such that $\phi(u_i) \to \infty$ as $|u_i| \to \infty$. For $u \in \mathbb{R}^K$ let $V(u) := \sum_{i=1}^K \phi_i(u_i)$ and $h(u) = \sum_{i=1}^K u_i$. Consider the problem

$$\min_u V(u) \text{ subject to } h(u) = c$$

   (a) Find the first order necessary condition for optimality using the Lagrangian $\hat{V}(u) = V(u) + p(c - h(u))$ with Lagrange multiplier p.
   Solution: The equation for a stationary point of $\hat{V}$ is $\nabla \hat{V} = 0$, which together with the constraint $h(u) = c$ leads to the following optimality conditions:

$$\phi_i'(u_i) = p \text{ for } 1 \leq i \leq K \qquad \sum_i u_i = c.$$

   (b) Let $v(c)$ denote the optimal value as a function of c, i.e. $v(c) = \min_{u:h(u)=c} V(u)$, and let $p(c)$ denote the value of the Lagrange multiplier found in part (a). Show that $v'(c) = p(c)$. In other words, the Lagrange multiplier is locally the ratio of change in optimal value to the change in the level c of the constraint. (Hint: To get started, fix a value of c and let u denote the corresponding optimal u vector and let $p = p(c)$. Changing c to $c + \delta c$ for $\delta c > 0$ results in a change of the optimal u_i to $u_i + \delta u_i$ such that $0 \leq \delta u_i \leq \delta c$ for each i. Apply Taylor's theorem.)
   Solution: Start as suggested. By the facts fact $\phi_i'(u_i) = p_i$ and $|\delta u_i| \leq \delta c$, Taylor's theorem implies $\phi_i(u_i + \delta u_i) = \phi_i(u_i) + p \cdot \delta u_i + o(\delta c)$, where $o(\delta c)/\delta c \to 0$ as $\delta c \to 0$. Thus,

$$\begin{aligned} v(c + \delta c) &= \sum_i \phi_i(u_i + \delta u_i) = \sum_i (\phi_i(u_i) + p\delta u_i + o(\delta c)) \\ &= v(c) + p \cdot \delta c + o(\delta c) \end{aligned}$$

   Equivalently, $v'(c) = p$.

2. [**Local but not global optimality of minimum principle solution**]
Consider the system/performance index

$$\dot{x} = u \qquad x(0) = 0 \qquad V(u) = \int_0^{t_1} \frac{u^2}{2} dt + \cos(x(t_1)).$$

where t_1 is a fixed terminal time and the terminal state $x(t_1)$ is freely varying.

(a) Find the optimality conditions implied by the minimum principle for this problem and simplify them as much as possible.

Solution: We apply Theorem 11.1 with $\ell(x,u,t) = \frac{u^2}{2}$, $f(x,u,t) = u$, $m(x) = \cos(x)$. We find $u^o(t) = \arg\min_u\{\frac{u^2}{2} + p(t)u\} = -p(t)$. The costate variable satisfies $\dot{p}(t) = -\nabla_x\{u^2 + p(t)u\} = 0$ so that p is constant in time (so we write "p" instead of "$p(t)$"). Using the boundary condition at t_1 for p we have $p = m'(x(t_1)) = -\sin(x(t_1))$. The control u is also constant in time and is given by $u = -p$. Therefore $x(t) = ut$. Thus, we can restrict the search for solutions to a search over the parameter p or over u or over $x(t_1)$. To be definite, we search over possible values x for $x(t_1)$. That is, we set $x(t_1) = x$, $u = x/t_1$, $p = -x/t_1$. Finding a solution comes down to finding x to satisfy $-\frac{x}{t_1} = -\sin(x)$ or $\frac{x}{t_1} = \sin(x)$.

(b) For what values of $t_1 > 0$ is there a unique solution and what is it? How does the number of solutions behave as $t_1 \to \infty$?

Solution: A simple sketch of $\frac{x}{t_1}$ and $\sin(x)$ shows that if $0 < t_1 \leq 1$ then there is a unique solution given by $x(t_1) = u = p = 0$, which has cost 2. Intuitively, even though the initial state is at a maximum point of m, it is too expensive to move away from it in a short amount of time to be worthwhile. For t_1 slightly larger than one there are three solutions. The number of solutions is nondecreasing with t_1 and for any odd positive integer k there is a value or range of t_1 so that there are precisely k solutions.

Another way to look at is is to note that the solutions satisfy either $x = 0$ or ($x \neq 0$ and $\frac{1}{t_1} = \frac{\sin(x)}{x} = \text{sinc}(x)$), and looking at a plot of $\text{sinc}(x)$ gives the same conclusions.

(c) Let $V(x,t_1)$ denote the minimum cost when the terminal state is x and the terminal time is t_1. Find $V(x,t_1)$ and use it to determine which of the solutions found in part (b) are optimal (Your answer should depend on t_1.)

Solution: We've found constant controls are optimal, so the minimum cost to reach x at time t_1 is obtained using the control $u = x/t_1$ so that

$$V(x,t_1) = \int_0^{t_1} \frac{1}{2}\left(\frac{x}{t_1}\right)^2 dt + \cos(x) = \frac{x^2}{2t_1} + \cos(x).$$

The solutions found in (b) correspond to the zeros of $\frac{\partial V}{\partial x}(x,t_1)$. As found above, there is only one solution if $0 < t_1 \leq 1$ corresponding to $x = 0$. For $t_1 > 1$ we find the second derivative of the cost at $x = 0$ is negative so that $x = 0$ is a local maximum and is thus not optimal. By the geometry of the situation we see that the two nonzero smallest magnitude solutions are the optimal ones for $t_1 > 1$. One of these increases from zero to π as t_1 increases from 1 to infinity, and the other is the negative of that one. There are no other solutions with $0 < x < 2\pi$. For any $x > 2\pi$, $V(x,t_1) > V(x-2\pi,t_1)$ so there are no global minima with $x \geq 2\pi$. Note

3. **[On the minimum principle with freely varying terminal time]**
Section 11.2 shows how to derive the minimum principle for t_1 fixed and $x(t_1)$ freely varying, which is stated as Theorem 11.1, by using Lagrange multipliers. Theorem 11.4 states a version of the minimum principle for t_1 freely varying and some of the indices of $x(t)$ specified. In this problem you are to explain how to derive Theorem 11.4 [1] by explaining how the derivation in

[1] Typos: p. 218, the equations in part (b): $x_i(t_i)$ should be $x_i(t_1)$. In the next line it should be for $j \in I^c$ not for $i \in I^c$

Section 11.2 (starting in the middle of page 208) should be modified. Steps 1-4 are identical except the function m has t as a second argument: $m(x(t_1), t)$.

(a) What modifications are needed in Step 5 to complete the derivation of Theorem 11.4?

Solution: The function η used to perturb x satisfies the constraint $\eta_i(t_1) = 0$ for $i \in I$ because $x_i(t_1)$ is constrained to equal x_{1i} for such i. We can still vary u enough between t_0 and t_1 to conclude that $\dot{p} = -\nabla_x H$ as before, but since only $\eta_j(t_1)$ for $j \in I^c$ are free, we get $p_j(t_1) = \frac{\partial m}{\partial x_j}(x^\circ(t_1), t_1)$ for $j \in I^c$. (There is also the constraint $x_i(t_1) = x_{1i}$ for $i \in I$ so there are still n constraints at t_1.)

Since the terminal time t_1 can be freely varied and we assume for this part that it is greater than t_0, we solve $\frac{d\hat{V}}{dt_1} = 0$. For the various terms that involve t_1 we have:

$$\frac{d}{dt_1}\left\{\int_{t_0}^{t_1} H(x^\circ, p, u^\circ, y)dt + \int_{t_0}^{t_1} \dot{p}^T x dt\right\} = H(x^\circ(t_1), p(t_1), u^\circ(t_1), t_1) + \dot{p}^T(t_1)x^\circ(t_1)$$

$$\frac{d}{dt_1}\left\{-p^T(t_1)x(t_1) + m(x(t_1), t_1)\right\}$$
$$= -\dot{p}^T(t_1)x(t_1) + \left\{\sum_{j\in I^c} \underbrace{\left(p_j(t_1) + \frac{\partial m}{\partial x_j}(x^\circ(t_1), t_1)\right)}_{0} \dot{x}_j(t_1)\right\} + \frac{\partial m}{\partial t}(x(t_1), t_1)$$

Adding the terms on the righthand sides gives (11.15):

$$\frac{\partial m}{\partial t}(x^\circ(t_1), t_1) + H(x^\circ(t_1), p(t_1), u^\circ(t_1), t_1) = 0.$$

(b) Consider Theorem 11.4 in the special case $I = \emptyset$; both t_1 and $x(t_1)$ are freely varying. The optimal terminal time t_1 could equal t_0 if running the system for a nonzero amount of time is more expensive than the decrease it brings in the terminal cost. Explain how the theorem can be extended to cover such case by finding a variation of (11.15) for the case $t_1 = t_0$. (Hint: Let $\bar{V}(\bar{u}, t_1)$ represent the cost as a function of t_1 if a constant control $\bar{u}$ is used. Then for $t_1 = t_0$ to be optimal it is necessary that $\frac{\partial \bar{V}}{\partial t_1}(\bar{u}, t_0) \geq 0$ for any $\bar{u}$.)

Solution: Let $t_1 = t_0 + \epsilon$. Assuming a constant control $\bar{u}$ is used over $[t_0, t_1]$, by Taylor's theorem:

$$\int_{t_0}^{t_0+\epsilon} \ell(x(t), \bar{u}, t)dt = \ell(x_0, \bar{u}, t_0)\epsilon + o(\epsilon)$$

$$x(t_0 + \epsilon) = x_0 + f(x_0, \bar{u}, t_0)\epsilon + o(\epsilon)$$

$$m(x(t_0 + \epsilon), t_0 + \epsilon) = m(x_0, t_0) + \frac{\partial m}{\partial t}(x_0, t_0)\epsilon + \frac{\partial m}{\partial x}(x_0, t_0)f(x_0, \bar{u}, t_0)\epsilon + o(\epsilon)$$

Therefore,

$$\frac{\bar{V}(\bar{u}, t_0 + \epsilon) - \bar{V}(\bar{u}, t_0)}{\epsilon} = \left[\frac{\partial m}{\partial t}(x_0, t_0) + H\left(x_0, \frac{\partial m}{\partial x}(x_0, t_0), \bar{u}, t_0\right)\right] + o(1)$$

or equivalently,

$$\frac{\partial \bar{V}}{\partial t_1}(\bar{u}, t_0) = \frac{\partial m}{\partial t}(x_0, t_0) + H\left(x_0, \frac{\partial m}{\partial x}(x_0, t_0), \bar{u}, t_0\right)$$

So a necessary condition for $t_1 = t_0$ to be optimal is:

$$\frac{\partial m}{\partial t}(x_0, t_0) + H\left(x_0, \frac{\partial m}{\partial x}(x_0, t_0), \bar{u}, t_0\right) \geq 0 \quad \text{for all } \bar{u}. \tag{1}$$

4. **[Optimal control with a free terminal time and state]**
As an application of the previous problem, consider the system/performance index

$$\dot{x} = u \qquad x(0) = x_0 \qquad V(u, t_1) = \int_0^{t_1} (1 + u^2)dt + \frac{B}{2}x^2(t_1)$$

where $B > 0$. As indicated in the notation, t_1 is variable so the problem is to find both the control u and t_1 to minimize $V(u, t_1)$. There is no constraint on the terminal state $x(t_1)$. For simplicity, assume $x_0 > 0$.

(a) Find the optimality conditions implied by the minimum principle for this problem.
Solution: We have $\ell(x, u, t) = 1 + u^2$, $m(x, t) = \frac{B}{2}x^2$, $f(x, u, t) = 1$.

$$\begin{aligned} u^\circ(t) &= \arg\min_u \{1 + u^2 + p(t)u\} = -\frac{p(t)}{2} \\ -\dot{p} &= \nabla_x \left\{1 + u^2 + pu\right\} = 0 \\ p(t_1) &= \nabla_x m(x(t_1), t_1) = Bx(t_1) \\ \text{so } p(t) &= Bx(t_1) \text{ for } t_0 \leq t \leq t_1 \end{aligned}$$

Hence both u and p are constant in time, with $u = -\frac{p}{2}$ and $p = Bx(t_1)$.
[Case 1: Solutions satisfying optimality necessary condition with $t_1 > 0$] Given a solution with $t_1 > 0$, the necessary condition for optimality (10.15) is $u^2(t_1) + 1 + p(t_1)u(t_1) = 0$. Substituting in $p = -2u$ gives the equation $1 - u^2 = 0$. From the geometry and the assumption $x_0 \geq 0$ we see that u should be negative, so $u = -1$. Therefore $p = 2$ and the final state is $x(t_1) = \frac{2}{B}$. Since $x(t) = x_0 + tu = x_0 - t$, this case makes sense only if $x_0 > \frac{2}{B}$, and in that case $t_1 = (x_0 - \frac{2}{B})$. The total minimum cost is $2t_1 + \frac{B}{2}x(t_1)^2 = 2x_0 - \frac{2}{B}$. (Note: If we seek a solution with $u = 1$ we get $x(t) = x_0 + t$ and $p = -2$ so the final state is $-\frac{2}{B}$, which is impossible. There is no solution with $u = 1$.) In summary, there exists a solution with $t_1 > 0$ satisfying the optimality condition only if $x_0 > \frac{2}{B}$ and in that case the optimal control is the constant control $u = -1$ with t_1 being the time $\frac{2}{B}$ is reached.
[Case 2: Solutions satisfying optimality necessary condition with $t_1 = 0$.] Since $\frac{\partial m}{\partial x}(x_0, t_0) = Bx_0$ the condition (1) becomes $1 + \bar{u}^2 + Bx_0\bar{u} \geq 0$ for all $\bar{u}$. The minimizing value of $\bar{u}$ is $\frac{-Bx_0}{2}$ so for $t_1 = t_0$ to be an optimal solution it is necessary that $1 - \left(\frac{Bx_0}{2}\right)^2 \geq 0$ or $0 \leq x_0 \leq \frac{2}{B}$.

(b) Under what conditions on x_0 and B is the optimal terminal time given by $t_1 = 0$?
Solution: We found in part (a) that for any $x_0 > 0$ there is a unique solution satisfying the necessary optimality conditions. The solution is $t_1 = 0$ if $0 < x_0 \leq \frac{2}{B}$.

(c) Explain what happens in the limit as $B \to \infty$? In particular, what is the limiting problem equivalent to and what is its solution?
Solution: Letting $B \to \infty$ corresponds imposing a terminal constraint $x(t_1) = 0$ while keeping the time t_1 free. Theorem 11.4 applies in this situation with $I = \{1\}$. Compared to the case above, where the terminal state was determined by the equation

$p(t_1) = \nabla_x m(x(t_1), t_1)$, the terminal state is instead constrained by $x(t_1) = 0$. The other optimality conditions are the same as above and imply the optimal control is constant with $u = 1$ and $p = 2$. So the optimal solution is to use the constant control $u = -1$ until state 0 is reached.

5. **[Stopping a pendulum in minimum time]**
Consider the system dynamics $\ddot{\theta}(t) = -\sin(\theta(t)) + \epsilon u(t)$ where ϵ is a small constant. It models the angle from vertical of a pendulum with the addition of a control. Consider the problem of selecting a control u such that $|u(t)| \leq 1$ for all $t \geq 0$ in order to minimize the time needed to reach the resting state $\theta(t_1) = \dot{\theta}(t_1) = 0$.

   (a) Letting $x_1 = \theta$ and $x_2 = \dot{\theta}$, write out the equations involving the state and co-state variables implied by the minimum principle. Identify the optimal control as a function of the co-state variables. (You don't need not solve the equations. Even with zero control the state trajectories of the pendulum can't be expressed in terms of elementary functions.)
   Solution: We have the constraint $x(t_1) = \vartheta$ while the terminal time t_1 is free and there is a hard constraint on the control. So this problem fits the setup of Theorem 11.4 and the optimality conditions there work except there is also a hard constraint on $u(t)$ for each t as in Theorem 11.2. We have

$$\begin{aligned} f(x,u) &= \begin{bmatrix} x_2 \\ -\sin(x_1) + \epsilon u \end{bmatrix} \\ \ell &= 1 \\ m &= 0 \\ H(x,u,p) &= 1 + p_1 x_2 + p_2(-\sin(x_1) + \epsilon u) \end{aligned}$$

   The optimal control is given by $u = \arg\min_{u:|u|\leq 1} H(x,u,p) = -\mathsf{sgn}(p_2)$. The equation (11.15) of the notes given $m = 0$ becomes $H(x(t_1), u(t_1), p(t_1)) = 0$ which together with $x(t_1) = \vartheta$ and the above equation for u gives $1 - \epsilon|p_2(t_1)| = 0$ or $p_2(t_1) = \pm\frac{1}{\epsilon}$.
   The dynamics $\dot{x} = f$ and $\dot{p} = -\nabla_x H$ then lead to the equations

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin(x_1) - \epsilon \cdot \mathsf{sgn}(p_2) \\ p_2 \cos(x_1) \\ -p_1 \end{bmatrix}$$

   with the boundary conditions $x(t_0) = x_o$, $x(t_1) = \vartheta$, and $p_2(t_1) = \pm\frac{1}{\epsilon}$. (The boundary conditions involve five degrees of freedom; one is due to the time t_1 being free.)

   (b) The total (kinetic plus potential) energy of the system, up to an additive constant, is $E(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 - \cos(\theta)$. Calculate $\frac{d}{dt}E(t)$ and then, based on your calculation, suggest a heuristic control.
   Solution: $\frac{d}{dt}E(t) = \epsilon u(t) x_2(t)$. This suggests the feedback control $u(t) = -\mathsf{sgn}(x_2(t))$ or, equivalently, $u(t) = -\mathsf{sgn}(\dot{\theta}(t))$. This makes intuitive sense – the control pushes in the opposite direction of the velocity. This is probably equal or very close to the optimal control when the system is far from equilibrium and ϵ is small. But for states close to equilibrium it is not optimal and can lead to the motion stopping at a state with $\theta(t) \neq 0$. It would be interesting to solve part (a) numerically and compare its performance to that of the heuristic control.