University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 12
Introduction to the Minimum Principle

Due: Wednesday, December 11, 11:59pm
Reading: Course notes, Chapter 11

1. [**Lagrange multipliers – sensitivity interpretation**]
Suppose ϕ_i for $1 \leq i \leq K$ are continuously differentiable strictly convex functions on the reals such that $\phi(u_i) \to \infty$ as $|u_i| \to \infty$. For $u \in \mathbb{R}^K$ let $V(u) := \sum_{i=1}^K \phi_i(u_i)$ and $h(u) = \sum_{i=1}^K u_i$. Consider the problem

$$\min_u V(u) \text{ subject to } h(u) = c$$

   (a) Find the first order necessary condition for optimality using the Lagrangian $\hat{V}(u) = V(u) + p(c - h(u))$ with Lagrange multiplier p.

   (b) Let $v(c)$ denote the optimal value as a function of c, i.e. $v(c) = \min_{u:h(u)=c} V(u)$, and let $p(c)$ denote the value of the Lagrange multiplier found in part (a). Show that $v'(c) = p(c)$. In other words, the Lagrange multiplier is locally the ratio of change in optimal value to the change in the level c of the constraint. (Hint: To get started, fix a value of c and let u denote the corresponding optimal u vector and let $p = p(c)$. Changing c to $c + \delta c$ for $\delta c > 0$ results in a change of the optimal u_i to $u_i + \delta u_i$ such that $0 \leq \delta u_i \leq \delta c$ for each i. Apply Taylor's theorem.)

2. [**Local but not global optimality of minimum principle solution**]
Consider the system/performance index

$$\dot{x} = u \qquad x(0) = 0 \qquad V(u) = \int_0^{t_1} \frac{u^2}{2} dt + \cos(x(t_1)).$$

where t_1 is a fixed terminal time and the terminal state $x(t_1)$ is freely varying.

   (a) Find the optimality conditions implied by the minimum principle for this problem and simplify them as much as possible.

   (b) For what values of $t_1 > 0$ is there a unique solution and what is it? How does the number of solutions behave as $t_1 \to \infty$?

   (c) Let $V(x, t_1)$ denote the minimum cost when the terminal state is x and the terminal time is t_1. Find $V(x, t_1)$ and use it to determine which of the solutions found in part (b) are optimal (Your answer should depend on t_1.)

3. [**On the minimum principle with freely varying terminal time**]
Section 11.2 shows how to derive the minimum principle for t_1 fixed and $x(t_1)$ freely varying, which is stated as Theorem 11.1, by using Lagrange multipliers. Theorem 11.4 states a version of the minimum principle for t_1 freely varying and some of the indices of $x(t)$ specified. In this problem you are to explain how to derive Theorem 11.4 [1] by explaining how the derivation in

[1] Typos: p. 218, the equations in part (b): $x_i(t_i)$ should be $x_i(t_1)$. In the next line it should be for $j \in I^c$ not for $i \in I^c$

Section 11.2 (starting in the middle of page 208) should be modified. Steps 1-4 are identical except the function m has t as a second argument: $m(x(t_1), t)$.

(a) What modifications are needed in Step 5 to complete the derivation of Theorem 11.4?

(b) Consider Theorem 11.4 in the special case $I = \emptyset$; both t_1 and $x(t_1)$ are freely varying. The optimal terminal time t_1 could equal t_0 if running the system for a nonzero amount of time is more expensive than the decrease it brings in the terminal cost. Explain how the theorem can be extended to cover such case by finding a variation of (11.15) for the case $t_1 = t_0$. (Hint: Let $\bar{V}(\bar{u}, t_1)$ represent the cost as a function of t_1 if a constant control $\bar{u}$ is used. Then for $t_1 = t_0$ to be optimal it is necessary that $\frac{\partial \bar{V}}{\partial t_1}(\bar{u}, t_0) \geq 0$ for any $\bar{u}$.)

4. **[Optimal control with a free terminal time and state]**
As an application of the previous problem, consider the system/performance index

$$\dot{x} = u \qquad x(0) = x_0 \qquad V(u, t_1) = \int_0^{t_1} (1 + u^2) dt + \frac{B}{2} x^2(t_1)$$

where $B > 0$. As indicated in the notation, t_1 is variable so the problem is to find both the control u and t_1 to minimize $V(u, t_1)$. There is no constraint on the terminal state $x(t_1)$. For simplicity, assume $x_0 > 0$.

(a) Find the optimality conditions implied by the minimum principle for this problem.

(b) Under what conditions on x_0 and B is the optimal terminal time given by $t_1 = 0$?

(c) Explain what happens in the limit as $B \to \infty$? In particular, what is the limiting problem equivalent to and what is its solution?

5. **[Stopping a pendulum in minimum time]**
Consider the system dynamics $\ddot{\theta}(t) = -\sin(\theta(t)) + \epsilon u(t)$ where ϵ is a small constant. It models the angle from vertical of a pendulum with the addition of a control. Consider the problem of selecting a control u such that $|u(t)| \leq 1$ for all $t \geq 0$ in order to minimize the time needed to reach the resting state $\theta(t_1) = \dot{\theta}(t_1) = 0$.

(a) Letting $x_1 = \theta$ and $x_2 = \dot{\theta}$, write out the equations involving the state and co-state variables implied by the minimum principle. Identify the optimal control as a function of the co-state variables. (You don't need not solve the equations. Even with zero control the state trajectories of the pendulum can't be expressed in terms of elementary functions.)

(b) The total (kinetic plus potential) energy of the system, up to an additive constant, is $E(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 - \cos(\theta)$. Calculate $\frac{d}{dt}E(t)$ and then, based on your calculation, suggest a heuristic control.