University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 11: Problems and Solutions
Linear Quadratic Regulators (LQR)

Due: Wednesday, December 4, 11:59pm
Reading: Course notes, Sections 10.3-10.7

1. **[LQR example]**
   Consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad x(0) = x_0$$

and infinite horizon cost $\int_0^\infty \|x\|^2 + ru^2 dt$, where $r > 0$.

(a) Determine what Theorem 10.6 implies about the LQR for this problem.

Solution: The system is in KCCF form and the uncontrollable matrix $A_c = [-1]$ is Hurwitz so the system is stabilizable (but not controllable). Also, $Q = I = C^T C$ for $C = I$ and A, C is observable. both parts of Theorem 10.6 apply and imply that the closed loop system matrix A_{cl} for the optimal feedback control is Hurwitz stable and the matrix $\bar{P}$ is positive definite.

(b) Compute by hand the LQR matrix $\bar{P}$, the optimal feedback control law, the closed loop state matrix A_{cl}, and the poles of the closed loop system.

Solution: Setting $\bar{P} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ the algebraic Riccati equation (ARE) becomes

$$\begin{bmatrix} 0 & 0 \\ a-b & b-c \end{bmatrix} + \begin{bmatrix} 0 & a-b \\ 0 & b-c \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{r} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} = 0$$

from which we find:

$$\bar{P} = \begin{bmatrix} \sqrt{r} & \frac{r}{1+\sqrt{r}} \\ \frac{r}{1+\sqrt{r}} & \frac{1}{2} + \frac{r}{1+\sqrt{r}} - \frac{r}{2(1+\sqrt{r})^2} \end{bmatrix}$$

$$\text{or } c = \frac{1 + 2\sqrt{r} + 2r + 2r\sqrt{r}}{2(1+\sqrt{r})^2}$$

$$u = -\frac{1}{r} B^T \bar{P} x = -\begin{bmatrix} \frac{1}{\sqrt{r}} & \frac{1}{1+\sqrt{r}} \end{bmatrix} x$$

$$A_{cl} = \begin{bmatrix} -\frac{1}{\sqrt{r}} & \frac{\sqrt{r}}{1+\sqrt{r}} \\ 0 & -1 \end{bmatrix}$$

The poles of the closed loop system are $-\frac{1}{\sqrt{r}}$ and -1. (Pole -1 is associated with the uncontrollable stable mode and is not moved by feedback.)

(c) Comment on how the feedback law and poles vary as r gets very large (expensive control) or very small (cheap control).

Solution: For r large the feedback multipliers get small and the controllable pole gets close to 0 (which is the open loop pole). For r small the first feedback multiplier gets large and the controllable pole gets large magnitude negative corresponding to a fast response time. However, the pole -1 is not controllable so the overall response time will be limited by that pole.

(d) Find the eigenvalues of the Hamiltonian matrix $\mathcal{H}$ by hand. Is your answer consistent with part (b)?

Solution:

$$\begin{aligned}\det(Is-\mathcal{H}) &= \det\begin{bmatrix} s & -1 & \frac{1}{r} & 0 \\ 0 & s+1 & 0 & 0 \\ 1 & 0 & s & 0 \\ 0 & 1 & 1 & s-1 \end{bmatrix} \\ &= s\det\begin{bmatrix} s+1 & 0 & 0 \\ 0 & s & 0 \\ 1 & 1 & s-1 \end{bmatrix} + \det\begin{bmatrix} -1 & \frac{1}{r} & 0 \\ s+1 & 0 & 0 \\ 1 & 1 & s-1 \end{bmatrix} \\ &= s^2(s^2-1) - (s^2-1)\frac{1}{r} = \left(s^2 - \frac{1}{r}\right)(s^2-1).\end{aligned}$$

The four eigenvalues of $\mathcal{H}$ are $\pm 1, \pm\frac{1}{\sqrt{r}}$, which is consistent with part (b) – they are the two stable eigenvalues found in part (b) and their negatives.

2. **[Variation of an LQR example]**
Consider the same linear system model as in Problem 1.

(a) Suppose the same cost function is used as in Problem 1 but with $\|x\|^2$ replaced by x_1^2. Explain how the LQR regulator and optimal cost are different from those found in Problem 1.

Solution: There is very little difference. The Q matrix is changed from the identity matrix to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so the corresponding C matrix becomes $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. The system remains detectable and observable and Theorem 10.6 applies as before. We find:

$$\bar{P} = \begin{bmatrix} \sqrt{r} & \frac{r}{1+\sqrt{r}} \\ \frac{r}{1+\sqrt{r}} & \frac{r}{1+\sqrt{r}} - \frac{r}{2(1+\sqrt{r})^2} \end{bmatrix}$$

The only difference is that the term $\frac{1}{2}$ is dropped from the bottom right entry of $\bar{P}$. In other words, the optimal cost $V^o(x) = x^T\bar{P}x$ is smaller by $\frac{1}{2}x_2(0)^2$. That is due to the fact that $\int_0^\infty (x^2(t))^2dt = \int_0^\infty (x_2(0)e^{-t})^2 dt = x_2^2(0)\int_0^\infty e^{-2t}dt = \frac{1}{2}x_2(0)^2$ no matter what control is used. The same feedback control is optimal and A_{cl} and the closed loop poles are the same as in Problem 1.

(b) The change in part (a) gives rise to a SISO system. Find the open loop transfer function $P(s) = C(Is-A)^{-1}B$ for part (a) and find the set of all solutions to the symmetric root locus equation (10.31) in the course notes, namely, $1 + \frac{1}{r}P(s)P(-s) = 0$. Do you recover the same two negative closed loop roots as before? Explain.

Solution: Computation as usual yields $P(s) = C(Is - A)^{-1}B = \frac{1}{s}$. At first this might seem surprising but since transfer functions don't take into account the uncontrollable subsystem we get the same transfer function as a pure integrator: $\dot{x_1} = u$. The symmetric root locus equation becomes $1 - \frac{1}{rs^2} = 0$ which has solutions $s = \pm\frac{1}{\sqrt{r}}$. Therefore this reflects only the controllable stable pole $-\frac{1}{\sqrt{r}}$ and its negative. It makes sense that since the system is not minimal (because it is not observable), the symmetric root locus equations would pertain to a minimal system model realization only.

(c) Suppose the same cost function is used as in Problem 1 but with $\|x\|^2$ replaced by x_2^2. Explain how the LQR regulator and optimal cost are different from those found in Problem 1. Find $\bar{P}$ and the optimal control for this variation.

Solution: There is a huge difference. The Q matrix is changed from the identity matrix to $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so the corresponding C matrix becomes $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$. The system is no longer observable and it is not even detectable, so Theorem 10.6 does not apply. Since the control cannot effect x_2 the optimal choice is to let $u(t) = 0$ for all t. We then see that $\bar{P} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. Also $A_{cl} = A$ so the closed loop poles are 0 and 1. A_{cl} is not Hurwitz. We see that $x_2(t) \to 0$ and $x_1(t) \to x_1(0) + x_2(0)/2$ as $t \to \infty$.

3. **[LQR with no control]**
Consider an LQR problem for an LTI system of the form $\dot{x} = Ax$ with cost $\int_0^\infty x^T Qx\, dt$ where $Q = C^T C$ for some matrix C. In other words, it is a general infinite horizon LQR problem except $B = 0$.

(a) Under what conditions on A and C does Theorem 10.6 part (i) apply and what does it imply under those conditions? Has the ARE appeared earlier in the course?

Solution: The system is stabilizable if and only if it is stable – meaning A is Hurwitz. If A is Hurwitz the system is also detectable. So part (i) of Theorem 10.6 applies if A is Hurwitz. If A is Hurwitz we can conclude there exists a positive semi-definite solution to the ARE which is unique in the class of positive semi-definite matrices. The ARE in this case is $A^T\bar{P} + \bar{P}A + Q = 0$, which is the Lyapunov stability equation encountered in Chapter 4. See a connection to the proof of Theorem 4.6.

(b) Under what conditions on A and C does Theorem 10.6 part (ii) apply and what does it imply under those conditions?

Solution: In order for Theorem 10.6 to apply we need to assume that A is Hurwitz (so (A, B) is detectable) and also that (A, C) is observable. In this case, in addition to the conclusions in part (i), we know that $\bar{P}$ is positive definite.

4. **[Stable subspace of the Hamiltonian matrix for LQR]**
Consider the LQR problem for the LTI system $\dot{x} = Ax + Bu$ and infinite horizon cost $\int_0^\infty x^T Qx + u^T Ru\, dt$ such that $Q = C^T C$. Assume R is positive definite, (A, B) is stabilizable, and (A, C) is detectable. Let $\bar{P}, A_{cl}$ and $\mathcal{H}$ be as in Chapter 10 of the notes.

(a) Show that $\mathcal{H}\begin{bmatrix} I \\ \bar{P} \end{bmatrix} = \begin{bmatrix} I \\ \bar{P} \end{bmatrix} A_{cl}$.

Solution: We need to show

$$\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ \bar{P} \end{bmatrix} = \begin{bmatrix} I \\ \bar{P} \end{bmatrix} A_{cl}$$

or equivalently

$$A - BR^{-1}B^T\bar{P} = A_{cl} \tag{1}$$
$$A^T\bar{P} + Q = -\bar{P}A_{cl} \tag{2}$$

Equation (1) gives the correct expression for A_{cl}; it is the closed loop matrix $A - BK$ for the optimal feedback control $u = -Kx$ for $K = R^{-1}B^T\bar{P}$. Substituting this expression for A_{cl} into (2) reduces the equation to the ARE.

(b) Show that $e^{\mathcal{H}t}\begin{bmatrix} I \\ \bar{P} \end{bmatrix} = \begin{bmatrix} I \\ \bar{P} \end{bmatrix} e^{A_{cl}t}$.

Solution: By part (a) and argument by induction on k, $\mathcal{H}^k\begin{bmatrix} I \\ \bar{P} \end{bmatrix} = \begin{bmatrix} I \\ \bar{P} \end{bmatrix} A_{cl}^k$ for $k \geq 0$. Multiplying both sides by $t^k/k!$ and summing over k from 0 to ∞ yields the equation to be shown.

(c) Part (b) shows that the columns of $\begin{bmatrix} I \\ \bar{P} \end{bmatrix}$ span the stable subspace of $\mathcal{H}$. This implies the following method for finding $\bar{P}$ using $\mathcal{H}$. First identify a $2n \times n$ matrix with columns that span the stable subspace of $\mathcal{H}$ and then do elementary column operations to make the upper half of the matrix the identity matrix. Then the bottom half will be $\bar{P}$. Illustrate this method by using it to find the $\bar{P}$ matrix $\bar{P} = [p]$ for the scalar LQR problem $\dot{x} = ax + bu$ with cost $\int_0^\infty qx^2 + ru^2\, dt$. Verify your answer by comparing to the solution of the ARE.

Solution: The characteristic polynomial of $\mathcal{H} = \begin{bmatrix} a & -b^2/r \\ -c^2 & -a \end{bmatrix}$ is $s^2 - a^2 - b^2q/r$, which has roots $\lambda_1 = -\sqrt{a^2 + b^2q/r}$ and $\lambda_2 = \sqrt{a^2 + b^2q/r}$. The eigenvector v^1 for the stable root λ_1 is determined by $(\lambda_1 I - \mathcal{H})v^1 = \vartheta$. Assuming v^1 is scaled so it has the form $v^1 = \begin{bmatrix} 1 \\ p \end{bmatrix}$ yields the equation $\begin{bmatrix} -\sqrt{a^2 + b^2q/r} - a & b^2/r \end{bmatrix}\begin{bmatrix} 1 \\ p \end{bmatrix} = 0$ or $p = \frac{r}{b^2}\left[\sqrt{a^2 + b^2c^2/r} + a\right]$. This is also the positive solution of the ARE which in this case reduces to $\frac{b^2p^2}{r} - 2ap - c^2 = 0$.

5. **[An observability equivalence]**
Let $Q = C^TC$. Show that (A, C) is an observable pair if and only if (A, Q) is an observable pair. (Hint: Apply the eigenvector criterion for observability – closely related to the Hautus Rosenbrock criterion.)

Solution: By the eigenvector criterion, (A, C) is observable if and only if $Cv \neq \vartheta$ for every eigenvector v of A and (A, C^TC) is observable if and only if $C^TCv \neq \vartheta$ for every eigenvector v of A. So it is enough to show that $Cv \neq \vartheta$ if and only if $C^TCv \neq \vartheta$ for any vector v or equivalently, $Cv = \vartheta$ if and only if $C^TCv = \vartheta$ for any vector v.

If $Cv = \vartheta$ then left multiplying both sides by C^T implies $C^TCv = \vartheta$. If $C^TCv = \vartheta$ then left multiplying both sides by v^T yields $v^TC^TCv = 0$ or equivalently $\|Cv\|^2 = 0$ which is equivalent to $Cv = \vartheta$.