University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 11
Linear Quadratic Regulators (LQR)

Due: Wednesday, December 4, 11:59pm
Reading: Course notes, Sections 10.3-10.7

1. **[LQR example]**
   Consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad x(0) = x_0$$

   and infinite horizon cost $\int_0^\infty \|x\|^2 + ru^2 dt$, where $r > 0$.

   (a) Determine what Theorem 10.6 implies about the LQR for this problem.

   (b) Compute by hand the LQR matrix $\bar{P}$, the optimal feedback control law, the closed loop state matrix A_{cl}, and the poles of the closed loop system.

   (c) Comment on how the feedback law and poles vary as r gets very large (expensive control) or very small (cheap control).

   (d) Find the eigenvalues of the Hamiltonian matrix $\mathcal{H}$ by hand. Is your answer consistent with part (b)?

2. **[Variation of an LQR example]**
   Consider the same linear system model as in Problem 1.

   (a) Suppose the same cost function is used as in Problem 1 but with $\|x\|^2$ replaced by x_1^2. Explain how the LQR regulator and optimal cost are different from those found in Problem 1.

   (b) The change in part (a) gives rise to a SISO system. Find the open loop transfer function $P(s) = C(Is - A)^{-1}B$ for part (a) and find the set of all solutions to the symmetric root locus equation (10.31) in the course notes, namely, $1 + \frac{1}{r}P(s)P(-s) = 0$. Do you recover the same two negative closed loop roots as before? Explain.

   (c) Suppose the same cost function is used as in Problem 1 but with $\|x\|^2$ replaced by x_2^2. Explain how the LQR regulator and optimal cost are different from those found in Problem 1. Find $\bar{P}$ and the optimal control for this variation.

3. **[LQR with no control]**
   Consider an LQR problem for an LTI system of the form $\dot{x} = Ax$ with cost $\int_0^\infty x^T Qx \, dt$ where $Q = C^T C$ for some matrix C. In other words, it is a general infinite horizon LQR problem except $B = 0$.

   (a) Under what conditions on A and C does Theorem 10.6 part (a) apply and what does it imply under those conditions? Has the ARE appeared earlier in the course?

   (b) Under what conditions on A and C does Theorem 10.6 part (b) apply and what does it imply under those conditions?

4. **[Stable subspace of the Hamiltonian matrix for LQR]**
Consider the LQR problem for the LTI system $\dot{x} = Ax + Bu$ and infinite horizon cost $\int_0^\infty x^T Q x + u^T R u \; dt$ such that $Q = C^T C$. Assume R is positive definite, (A, B) is stabilizable, and (A, C) is detectable. Let $\bar{P}$, A_{cl} and $\mathcal{H}$ be as in Chapter 10 of the notes.

   (a) Show that $\mathcal{H} \begin{bmatrix} I \\ \bar{P} \end{bmatrix} = \begin{bmatrix} I \\ \bar{P} \end{bmatrix} A_{cl}$.

   (b) Show that $e^{\mathcal{H}t} \begin{bmatrix} I \\ \bar{P} \end{bmatrix} = \begin{bmatrix} I \\ \bar{P} \end{bmatrix} e^{A_{cl}t}$.

   (c) Part (b) shows that the columns of $\begin{bmatrix} I \\ \bar{P} \end{bmatrix}$ span the stable subspace of $\mathcal{H}$. This implies the following method for finding $\bar{P}$ using $\mathcal{H}$. First identify a $2n \times n$ matrix with columns that span the stable subspace of $\mathcal{H}$ and then do elementary column operations to make the upper half of the matrix the identity matrix. Then the bottom half will be $\bar{P}$. Illustrate this method by using it to find the $\bar{P}$ matrix $\bar{P} = [p]$ for the scalar LQR problem $\dot{x} = ax + bu$ with cost $\int_0^\infty qx^2 + ru^2 \; dt$. Verify your answer by comparing to the solution of the ARE.

5. **[An observability equivalence]**
Let $Q = C^T C$. Show that (A, C) is an observable pair if and only if (A, Q) is an observable pair. (Hint: Apply the eigenvector criterion for observability – closely related to the Hautus Rosenbrock criterion.)