**University of Illinois** **Fall 2024**

# ECE 515/ME 540: Problem Set 10: Problems and Solutions
# Dynamic programming and the HJB equation

**Due:** Wednesday, November 20, 11:59pm
**Reading:** Course notes, Sections 10.1-10.2

1. **[A discrete time and space dynamic programming problem]**
Consider the transition cost matrix $L$ with entries $L(i,j)$ and terminal cost vector $m$ with entries $m(i)$ for $i,j$ in the state space $S = \{1,2,3,4\}$:

$$L = \begin{bmatrix} 4 & 4 & \infty & \infty \\ \infty & 2 & 0 & \infty \\ \infty & 0 & 4 & 4 \\ 4 & \infty & \infty & 10 \end{bmatrix} \qquad m = \begin{bmatrix} 40 \\ 30 \\ 20 \\ 10 \end{bmatrix}$$

The cost of a trajectory $x_0, x_1, \ldots, x_T$ in the state space is $\sum_{t=0}^{T-1} L(x_t, x_{t+1}) + m(x(T))$. Let $V^\circ(i,t)$ be the minimum cost over all trajectories of length $T-t$ that start in state $i$ at time $t$, for $i \in S$ and $t \in \{0,1,\ldots,T\}$.

   (a) Write out the discrete time dynamic programming equations for determining $V^\circ$ from $L$ and $m$.

   **Solution:** $V^\circ(i,t-1) = \min_{j\in S} L(i,j) + V^\circ(j,t)$ for $t \le T$ with the boundary condition $V^\circ(i,T) = m(i)$.

   (b) Find $V^\circ(j,t)$ for $j \in S$ and $0 \le t \le 10$ for $T = 10$.

   **Solution:** $$V^\circ = \begin{bmatrix} 20 & 18 & 20 & 18 & 20 & 18 & 20 & 18 & 24 & 34 & 40 \\ 14 & 16 & 14 & 16 & 14 & 16 & 14 & 16 & 14 & 20 & 30 \\ 16 & 14 & 16 & 14 & 16 & 14 & 16 & 14 & 18 & 14 & 20 \\ 22 & 24 & 22 & 24 & 22 & 24 & 22 & 28 & 30 & 20 & 10 \end{bmatrix}.$$

   (c) What is the minimum cost trajectory from state 4 at time 0 to state 4 at time 10, again assuming $T = 10$ and including the terminal cost?

   **Solution:** 41232323234

   (d) Is $V^\circ(j,0)$ bounded as $T \to \infty$? Explain why.

   **Solution:** Yes. Trajectories can cycle between states 2 and 3 with zero cost. In fact we see that a pattern emerges and the columns are alternating for $t \le T-4$. So $V^\circ(j,t) \le 24$ for all $t \le T-4$.

2. **[A simple optimization problem]**
Consider the LTI system and cost function:

$$\dot{x} = u \qquad x(0) = 0 \qquad V(u) = \int_0^T u^4(\tau)d\tau + (x(T) - b)^4$$

where $b$ and $T$ are known constants with $T > 0$.

   (a) Find the minimum cost assuming that the control $u$ is constant in time: $u(\tau) = \bar{u}$ for $0 \le \tau \le T$ where $\bar{u}$ should be selected depending on $b$ and $T$ to minimize the cost.

**Solution:** $\bar{V} = \min_{\bar{u}} T\bar{u}^4 + (T\bar{u} - b)^4$. The optimal choice of $\bar{u}$ is $\frac{b}{1+T}$ yielding cost $\bar{V} = \frac{b^4}{(1+T)^3}$.

(b) It can be shown using Jensen's inequality that constant controls are optimal. Using that fact and the answer to part (a), derive an expression for the value function $V^\circ(x,t)$ for $x \in \mathbb{R}$ and $t \leq T$.

**Solution:** $V^\circ(t,t)$ is equal to the minimum cost for the original problem with $T$ replaced by $T-t$ and $b$ replaced by $b-x$. Therefore, $V^\circ(x,t) = \frac{(b-x)^4}{(1+T-t)^3}$ and the optimal control is given by $u = \frac{b-x}{1+T-t}$.

(c) Write out the HJB equation including the boundary condition and verify that $V^\circ$ as found in part (b) is a solution. This gives a second proof that constant controls are optimal for this problem.

**Solution:** The HJB equation is

$$-\frac{\partial V^\circ}{\partial t} = \min_u \left\{ u^4 + \frac{\partial V^\circ}{\partial x} u \right\} \qquad V^\circ(x,T) = (x-b)^4$$

Towards verifying that $V^\circ$ from part (c) is a solution we check that it satisfies the boundary condition and we find the minimum of the Hamiltonian: $\min_u\{u^4 + pu\} = -3\left(\frac{p}{4}\right)^{4/3}$ where the minimizer is $u = -\left(\frac{p}{4}\right)^{\frac{1}{3}}$. So the HJB equation becomes $-\frac{\partial V^\circ}{\partial t} = -3\left(\frac{1}{4}\frac{\partial V^\circ}{\partial x}\right)^{4/3}$ which is readily verified.

3. **[HJB for an infinite horizon optimal control problem]**
Problem 10.7.1 of the course notes.

**Solution:** (a) To avoid trivialities we assume that for any $x_0$ there is a control such that the cost is finite. In that case the derivation of the HJB equation in the notes goes through with $t_1 = \infty$. Theorem 10.1 still holds.

(b) Since the function $f$ and running cost function $\ell$ are not time dependent the value function $V^\circ(x,t)$ will not depend on $t$. So we can write $V^\circ(x,t) = J^\circ(x)$ and in the HJB equation (10.5) in the notes, $\frac{\partial V^\circ}{\partial t} = 0$. The HJB equation thus becomes

$$\min_u \left[ \ell(x,u) + \frac{dJ^\circ(x)}{dx} f(x,u) \right] = 0 \qquad (1)$$

Minimizing with respect to $u$ we get the simultaneous equations for $u$ and $\frac{dJ^\circ(x)}{dx}$ :

$$\ell(x,u) + \frac{dJ^\circ(x)}{dx} f(x,u) = 0$$
$$\frac{\partial \ell(x,u)}{\partial u} + \frac{dJ^\circ(x)}{dx}\frac{\partial f(x,u)}{\partial u} = \vartheta_{1\times m}$$

Of course setting the gradient with respect to $u$ to zero could pick out a local maximum or critical point – not necessarily a minimum, so we may have introduced multiple solutions to sort out later.

(c) For the simple integrator problem, $f(x,u) = u$ and $\ell(x,u) = u^2 + x^4$. The above equations become

$$u^2 + x^4 + \frac{dJ^\circ(x)}{dx} u = 0$$
$$2u + \frac{dJ^\circ(x)}{dx} = 0$$

The second equation gives $\frac{dJ^\circ(x)}{dx} = -2u$. Substituting into the first equation gives $x^4 - u^2 = 0$. From the geometry of the problem we take the solution $u = -\mathsf{sgn}(x)x^2$. This gives the equation

$$\frac{dJ^\circ(x)}{dx} = 2\mathsf{sgn}(x)x^2.$$

Since $J^\circ(0) = 0$ we can integrate to find $J^\circ(x) = \frac{2}{3}|x^3|$.
Let's check to make sure this choice satisfies the HJB equation (1). It becomes

$$\min_u \left[u^2 + x^4 + 2x^2\mathsf{sgn}(x)u\right] = 0$$

which is indeed satisfied.
In summary, the optimal control has the feedback form: $u = -\mathsf{sgn}(x)x^2$ and the minimum cost for initial state $x$ is $J^\circ(x) = \frac{2}{3}|x^3|$.