University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 10
Dynamic programming and the HJB equation

Due: Wednesday, November 20, 11:59pm
Reading: Course notes, Sections 10.1-10.2

1. **[A discrete time and space dynamic programming problem]**
Consider the transition cost matrix L with entries $L(i,j)$ and terminal cost vector m with entries $m(i)$ for i,j in the state space $S = \{1,2,3,4\}$:

$$L = \begin{bmatrix} 4 & 4 & \infty & \infty \\ \infty & 2 & 0 & \infty \\ \infty & 0 & 4 & 4 \\ 4 & \infty & \infty & 10 \end{bmatrix} \qquad m = \begin{bmatrix} 40 \\ 30 \\ 20 \\ 10 \end{bmatrix}$$

The cost of a trajectory $x_0, x_1, \ldots, x_T$ in the state space is $\sum_{t=0}^{T-1} L(x_t, x_{t+1}) + m(x(T))$. Let $V^\circ(i,t)$ be the minimum cost over all trajectories of length $T-t$ that start in state i at time t, for $i \in S$ and $t \in \{0,1,\ldots,T\}$.

   (a) Write out the discrete time dynamic programming equations for determining V° from L and m.

   (b) Find $V^\circ(j,t)$ for $j \in S$ and $0 \le t \le 10$ for $T = 10$.

   (c) What is the minimum cost trajectory from state 4 at time 0 to state 4 at time 10, again assuming $T = 10$ and including the terminal cost?

   (d) Is $V^\circ(j,0)$ bounded as $T \to \infty$? Explain why.

2. **[A simple optimization problem]**
Consider the LTI system and cost function:

$$\dot{x} = u \qquad x(0) = 0 \qquad V(u) = \int_0^T u^4(\tau)d\tau + (x(T) - b)^4$$

where b and T are known constants with $T > 0$.

   (a) Find the minimum cost assuming that the control u is constant in time: $u(\tau) = \bar{u}$ for $0 \le \tau \le T$ where $\bar{u}$ should be selected depending on b and T to minimize the cost.

   (b) It can be shown using Jensen's inequality that constant controls are optimal. Using that fact and the answer to part (a), derive an expression for the value function $V^\circ(x,t)$ for $x \in \mathbb{R}$ and $t \le T$.

   (c) Write out the HJB equation including the boundary condition and verify that V° as found in part (b) is a solution. This gives a second proof that constant controls are optimal for this problem.

3. **[HJB for an infinite horizon optimal control problem]**
Problem 10.7.1 of the course notes.