**University of Illinois** **Fall 2024**

# ECE 515/ME 540: Problem Set 9: Problems and Solutions
# Frequency domain stabilization / review of earlier material

**Due:** Wednesday, November 6, 11:59pm
**Reading:** Course notes, Chapter 8.2, 9 (also review Chapters 1-8.1)

1. **[Frequency domain approach to stabilize harmonic oscillator]**
Consider the harmonic oscillator with position measurements given in state space form by:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

We found in problem 4 of problem set 7 that using control $u = -K\hat{x}$ with the observer given by $\dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly$ with $K = \begin{bmatrix} 3 & 4 \end{bmatrix}$ and $L = \begin{bmatrix} 16 \\ 63 \end{bmatrix}$ places the poles of the closed loop system at -2,-2, -8, -8.

(a) Find the transfer function $P(s)$ of the plant $A, B, C$. (Hint: It is given by $P(s) = C(Is - A)^{-1}B$.)
**Solution:** $P(s) = \frac{1}{s^2+1}$.

(b) Find the transfer function $G(s)$ of the stabilizing controller described above – with input $y$ and output $-u$.
**Solution:**

$$G(s) = K(Is - (A - KB - LC))^{-1}L$$
$$= \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} s+16 & -1 \\ 67 & 6+4 \end{bmatrix}^{-1} \begin{bmatrix} 16 \\ 63 \end{bmatrix} = \frac{300s + 125}{s^2 + 20s + 131}$$

(c) By our design, using this controller as pictured in Fig. 8.1 of the notes results in a stable system so that $y$ will converge to zero. Now consider an input $r(t)$ with Laplace transform $R(s)$ that is added to the plant input as a perturbation of $U$ so that the closed loop transfer function for $R$ to $Y$ is given by $P_{cl}(s) = \frac{P(s)}{1+G(s)P(s)}$. Rederive your answer to part (b) by using the frequency domain method of Section 8.2. That is, write $G(s) = \frac{n(s)}{d(s)}$ and solve the equations for $n(s)$ and $d(s)$ that arise from making the poles of $P_{cl}$ equal to -2,-2,-8,-8.
**Solution:** Writing the plant transfer function as $P(s) = \frac{b(s)}{a(s)}$ where $b(s) = 1$ and $a(s) = s^2 + 1$, $P_{cl}(s)$ can be expressed as:

$$P_{cl}(s) = \frac{b(s)d(s)}{a(s)d(s) + b(s)n(s)}.$$

We want the denominator of $P_{cl}(s)$ equal to $(s + 2)^2(s + 8)^2$. Letting $n(s) = n_1 s + n_2$ and $d(s) = s^2 + d_1 s + d_2$ yields

$$(s^2 + 1)(s^2 + d_1 s + d_2) + n_1 s + n_2 = (s + 2)^2(s + 8)^2$$

or

$$s^4 + d_1 s^3 + (d_2+1)s^2 + (d_1+n_1)s + d_2 + n_2 = s^4 + 20s^3 + 132s^2 + 320s + 256$$

which yields $n_1 = 300, n_2 = 125, d_1 = 20, d_2 = 132$ so that $G(s)$ is again found to be the same as found in part (b).

2. **[BIBO stability problem]**
Consider the following third order state space model

$$\dot{x} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -4 & 5 \\ 0 & 0 & -6 \end{bmatrix} x + Bu$$
$$y = Cx + Du$$

such that the matrices $B, C, D$ have dimensions $3 \times m$, $p \times 3$ and $p \times m$, respectively, where $m$ and $p$ can be greater than one (i.e. MIMO case).

(a) Under what conditions on the $B, C$ and $D$ matrices is the system BIBO stable? Make your answer as explicit as possible.
**Solution:** The $A$ matrix is upper triangular so its eigenvalues are along the diagonal. Eigenvalues -4 and -6 are stable so the system will be BIBO stable if and only if the mode corresponding to eigenvalue 1 is either uncontrollable or unobservable or both. There is no restriction on $D$. So we seek to find the right eigenvector $v^1$ and left eigenvector $r^{1*}$ for eigenvalue 1. That is, solve

$$\begin{bmatrix} 0 & 2 & -3 \\ 0 & 5 & -5 \\ 0 & 0 & 7 \end{bmatrix} v^1 = \vartheta \quad \text{and} \quad r^{1*} \begin{bmatrix} 0 & 2 & -3 \\ 0 & 5 & -5 \\ 0 & 0 & 7 \end{bmatrix} = \vartheta$$

Up to scalar constant multiples, we find $v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $r^{1*} = \begin{bmatrix} 1 & -2/5 & 1/7 \end{bmatrix}$. So the system is BIBO stable if and only if ($r^{1*}B = \vartheta$ or $Cv_1 = \vartheta$).

(b) Under what conditions on the $B, C$ and $D$ matrices does there exist a state feedback $u = -Kx + r$ such that the closed loop system mapping $r$ to $y$ is BIBO stable?
**Solution:** State feedback can achieve BIBO stability for any $A, B, C, D$. Here is a proof. By using a state space transformation if necessary we can assume the system is in the KCCF form:

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} A_c & A_{1,2} \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + Du$$

Then using state feedback of the form $u = -[K_c \;\; 0] \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + r$ in the original system will yield the same input-output response (for zero initial conditions) as the reduced order system $A_c, B_c, C_c, D$ with state $x_c$ and state feedback $u = -K_c x_c + r$. Since the reduced order system is controllable the state feedback can be used to place the poles arbitrarily and therefore to stabilize the system (i.e. make $A_c - B_c K_c$ Hurwitz) yielding BIBO stability. The control does not influence $x_{\bar{c}}$ so that $x_{\bar{c}}(t) = 0$ for all $t$ for any control.

3. **[On stability of a predator prey model]**
Consider the following Lotka-Volterra type predator prey model where $\alpha$ is a positive constant and we consider states with $x_1 \geq 0$ and $x_2 \geq 0$ :

$$\dot{x_1} = x_1(\alpha - x_2)$$
$$\dot{x_2} = x_2(x_1 - 1)$$

(a) Which coordinate of $x$ represents the size or density of the predator population and which coordinate represents the size or density of the prey population? Explain.

**Solution:** $x_1$ represents prey population which increases as long as the size of the predator population is less than $\alpha$, whereas the predator population grows when the prey population is greater than one.

(b) Find all equilibrium points. For each equilibrium point find the linear system approximation of the dynamics in a neighborhood of the equilibrium and find whether Lyapunov's first method (i.e. looking at stability of the linear system) implies any stability property of the nonlinear system for each equilibrium point.

**Solution:** Let $F(x) = \begin{bmatrix} x_1(\alpha - x_2) \\ x_2(x_1 - 1) \end{bmatrix}$ so the dynamical system is $\dot{x} = F(x)$. Equilibrium points are solutions to $F(x_e) = 0$. At equilibrium, ($x_1$=0 or $x_2 = \alpha$) and ($x_2 = 0$ or $x_1 = 1$). Hence there are two equilibrium points, $\vartheta$ and $x_e = \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$.

Note that the derivative (aka Jacobian matrix) of $F$ at a point $x$ is given by

$$\frac{\partial F}{\partial x}(x) = \begin{bmatrix} \alpha - x_2 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix}.$$

**Equilibrium point** $\vartheta$. For equilibrium point $\vartheta$ we write $x = \vartheta + \delta x = \delta x$. The derivative of $F$ at $\vartheta$ is $\begin{bmatrix} \alpha & 0 \\ 0 & -1 \end{bmatrix}$ so ignoring higher order terms

$$\dot{\delta x} = \begin{bmatrix} \alpha & 0 \\ 0 & -1 \end{bmatrix} \delta x$$

This has modal form with a positive eigenvalue $\alpha$ for $\delta x_1$ and negative eigenvalue for $\delta x_2$. So near $\vartheta$ the prey population grows while the predator population shrinks. Since the linear system has an eigenvalue with strictly positive real part we conclude that $\vartheta$ is an unstable equilibrium for the nonlinear model.

**Equilibrium point** $x_e = \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$. For equilibrium point $x_e$ we write $x = x_e + \delta x$. The derivative of $F$ at $x_e$ is $\begin{bmatrix} 0 & -1 \\ \alpha & 0 \end{bmatrix}$ so ignoring higher order terms

$$\dot{\delta x} = \begin{bmatrix} 0 & -1 \\ \alpha & 0 \end{bmatrix} \delta x$$

The eigenvalues of the $A$ matrix are purely imaginary $\pm j\sqrt{\alpha}$. While this implies that the linear system is Lyapunov stable and not asymptotically stable, we can't make any definitive conclusion about stability of $x_e$ for the nonlinear system.

(c) Can you deduce and additional stability property for the equilibrium points you found by using the function $V(x) = x_1 - \ln(x_1) + x_2 - \alpha \ln(x_2)$ defined on the open quadrant $\{x_1 > 0, x_2 > 0\}$?

**Solution:** Note that $\frac{d}{dt}V(x_t) = \nabla V \cdot F = \begin{bmatrix} 1 - \frac{1}{x_1} \\ 1 - \frac{\alpha}{x_2} \end{bmatrix} \cdot F(x) = 0$. Also, $V$ is a separable function and the component functions $x_1 - \ln(x_1)$ and $x_2 - \alpha \ln(x_2)$ are both strictly convex over $(0, +\infty)$ and converge to $+\infty$ at 0 and $+\infty$. Since $\nabla V(x_e) = \vartheta$, $V$ is minimized at $x_e$. So $\tilde{V}$ defined by $\tilde{V}(x) = V(x) - V(x_e)$ is a Lyapunov function for the dynamics at $x_e$ from which we can conclude that $x_e$ is a Lyapunov stable equilibrium point for the nonlinear system. Moreover, since $\frac{d}{dt}V(x_t) = 0$ and given the two dimensional structure we can see that the trajectories of $x$ are periodic and go around the equilibrium point in the counter-clockwise direction. It follows that $x_e$ is not asymptotically stable.

4. **[Hermitian symmetric matrices]**
Suppose $Q$ is an $n \times n$ Hermitian symmetric matrix, meaning $Q = Q^*$.

(a) Show that the eigenvalues of $Q$ are real valued. (Hint: For any vector $v$, $v^*Qv$ is real valued because it is equal to its complex conjugate.)

**Solution:** If $\lambda$ is an eigenvector of $Q$ there exists a corresponding eigenvector $v$ so $Qv = \lambda v$ and $v^*Qv = \lambda v^*v = \lambda\|v\|^2$. Thus $\lambda = v^*Qv/\|v\|^2$, which is the ratio of two real numbers, so $\lambda$ is real as well.

(b) Show that the eigenvectors of $Q$ for different eigenvalues are orthogonal.

**Solution:** Suppose $v^i$ is an eigenvector for eigenvalue $\lambda_i$ for $i = 1, 2$. Then $Qv^i = \lambda_i v^i$ and by part (a), $v^{i*}Q = \lambda_i v^{i*}$. Thus $v^{1*}Qv^2 = v^{1*}(Qv^2) = \lambda_2 v^{1*}v^2$ and $v^{1*}Qv^2 = (v^{1*}Q)v^2 = \lambda_1 v^{1*}v^2$. So $(\lambda_1 - \lambda_2)v^{1*}v^2 = 0$. Therefore, if $\lambda_1 \neq \lambda_2$ then $\langle v_1, v_2 \rangle = v^{1*}v^2 = 0$.

(c) Let $V$ be an $n \times n$ matrix with columns being an orthonormal basis of eigenvectors of $Q$ (can be shown to exist) and let $\Lambda$ be the diagonal matrix with the corresponding eigenvalues down the diagonal. Show that $Q = V\Lambda V^*$.

**Solution:** Note that $V^*V = I$, or in other words, $V^* = V^{-1}$ (i.e. $V$ is a unitary matrix). The eigenvalue-eigenvector relationship can be written as $QV = V\Lambda$. Multiplying by $V^*$ on the right gives the required equality.