**University of Illinois** **Fall 2024**

# ECE 515/ME 540: Problem Set 9
## Frequency domain stabilization / review of earlier material

**Due:** Wednesday, November 6, 11:59pm
**Reading:** Course notes, Chapter 8.2, 9 (also review Chapters 1-8.1)

1. **[Frequency domain approach to stabilize harmonic oscillator]**
Consider the harmonic oscillator with position measurements given in state space form by:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

We found in problem 4 of problem set 7 that using control $u = -K\hat{x}$ with the observer given by $\dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly$ with $K = \begin{bmatrix} 3 & 4 \end{bmatrix}$ and $L = \begin{bmatrix} 16 \\ 63 \end{bmatrix}$ places the poles of the closed loop system at -2,-2, -8, -8.

   (a) Find the transfer function $P(s)$ of the plant $A, B, C$. (Hint: It is given by $P(s) = C(Is - A)^{-1}B$.)

   (b) Find the transfer function $G(s)$ of the stabilizing controller described above – with input $y$ and output $-u$.

   (c) By our design, using this controller as pictured in Fig. 8.1 of the notes results in a stable system so that $y$ will converge to zero. Now consider an input $r(t)$ with Laplace transform $R(s)$ that is added to the plant input as a perturbation of $U$ so that the closed loop transfer function for $R$ to $Y$ is given by $P_{cl}(s) = \frac{P(s)}{1+G(s)P(s)}$. Rederive your answer to part (b) by using the frequency domain method of Section 8.2. That is, write $G(s) = \frac{n(s)}{d(s)}$ and solve the equations for $n(s)$ and $d(s)$ that arise from making the poles of $P_{cl}$ equal to -2,-2,-8,-8.

2. **[BIBO stability problem]**
Consider the following third order state space model

$$\dot{x} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -4 & 5 \\ 0 & 0 & -6 \end{bmatrix} x + Bu$$
$$y = Cx + Du$$

such that the matrices $B, C, D$ have dimensions $3 \times m$, $p \times 3$ and $p \times m$, respectively, where $m$ and $p$ can be greater than one (i.e. MIMO case).

   (a) Under what conditions on the $B, C$ and $D$ matrices is the system BIBO stable? Make your answer as explicit as possible.

   (b) Under what conditions on the $B, C$ and $D$ matrices does there exist a state feedback $u = -Kx + r$ such that the closed loop system mapping $r$ to $y$ is BIBO stable?

3. **[On stability of a predator prey model]**
Consider the following Lotka-Volterra type predator prey model where $\alpha$ is a positive constant and we consider states with $x_1 \geq 0$ and $x_2 \geq 0$ :

$$\dot{x_1} = x_1(\alpha - x_2)$$
$$\dot{x_2} = x_2(x_1 - 1)$$

   (a) Which coordinate of $x$ represents the size or density of the predator population and which coordinate represents the size or density of the prey population? Explain.

   (b) Find all equilibrium points. For each equilibrium point find the linear system approximation of the dynamics in a neighborhood of the equilibrium and find whether Lyapunov's first method (i.e. looking at stability of the linear system) implies any stability property of the nonlinear system for each equilibrium point.

   (c) Can you deduce and additional stability property for the equilibrium points you found by using the function $V(x) = x_1 - \ln(x_1) + x_2 - \alpha \ln(x_2)$ defined on the open quadrant $\{x_1 > 0, x_2 > 0\}$?

4. **[Hermitian symmetric matrices]**
Suppose $Q$ is an $n \times n$ Hermitian symmetric matrix, meaning $Q = Q^*$.

   (a) Show that the eigenvalues of $Q$ are real valued. (Hint: For any vector $v$, $v^*Qv$ is real valued because it is equal to its complex conjugate.)

   (b) Show that the eigenvectors of $Q$ for different eigenvalues are orthogonal.

   (c) Let $V$ be an $n \times n$ matrix with columns being an orthonormal basis of eigenvectors of $Q$ (can be shown to exist) and let $\Lambda$ be the diagonal matrix with the corresponding eigenvalues down the diagonal. Show that $Q = V\Lambda V^*$.