University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 8: Problems and Solutions
Feedback: Tracking and Disturbance Rejection

Due: Wednesday, October 30, 11:59pm
Reading: Course notes, Chapter 8 (also review Chapter 7)

1. **[Controllability indices]**
Consider the controllability matrix $\mathcal{C} = [B\ AB\ A^2B\ \cdots\ A^{n-1}B]$ for a linear system model, where A is $n \times n$ and B is $n \times m$ for positive integers m, n. Let $b^1, \ldots, b^m$ denote the columns of B. Let $\bar{\mathcal{C}}$ be obtained by reordering the columns of $\mathcal{C}$ as follows (such reordering does not change the column span):

$$\overline{\mathcal{C}} = \left[b^1\ Ab^1\ A^2b^1\ \cdots\ A^{n-1}b^1\ b^2\ Ab^2\ \cdots\ A^{n-1}b^2\ b^3\ \ \cdots\ A^{n-1}b^m\right].$$

A basis for the column span of $\mathcal{C}$, or equivalently, the range space of $\mathcal{C}$, can be found by the following algorithm. Consider the columns of $\overline{\mathcal{C}}$ one by one from left to right, and add any column to the basis that is not in the span of the columns before it.

   (a) Show that whenever a column of the form A^jb^i is not included in the basis then any column of the form $A^{j'}b^i$ with $j' > j$ will not be included. (Hint: Start by considering the columns with $i = 1$.)

   Solution: Suppose there exists j_1 such that $A^{j_1}b^1$ is the first column not selected by the algorithm. (If not, the algorithm will terminate before considering any columns of the form $A^{j_1}b^i$ with $i \geq 2$.) We will show by induction on j that $A^jb^1 \in \text{span}\{b^1, \ldots, A^{j-1}b^1\}$ for $j_1 \leq j \leq n-1$, and hence none of those columns will be selected by the algorithm. The base case $j = j_1$ is true by the definition of j_1. For the induction step, suppose that for some j with $j_1 \leq j \leq n-2$ that $A^jb^1 \in \text{span}\{b^1, Ab^1, \ldots, A^{j-1}b^1\}$ Then $A^{j+1}b^1 = A(A^jb^1) \in A\text{span}\{b^1, Ab^1, \ldots, A^{j-1}b^1\} \subset \text{span}\{b^1, Ab^1, \ldots, A^jb^1\}$ so the statement is true for $j+1$ and the proof by induction is complete. So far we have proved that once a column of the form A^jb^1 is not included in the basis by the algorithm then no other columns of that form are added. After finishing with the columns of the form A^jb^1 the span, Σ_1, of the columns selected up to that point is invariant under multipliction by A. In other words, $A\Sigma_1 \subset \Sigma_1$.

   Using a similar argument as above we can show by induction on the index i of the columns of B that if any column of the form A^jb^i is not included then the algorithm will not select any more columns with the same b^i and just after that, the span of the space of columns σ_i selected up to that point is invariant under A.

   (b) Let μ_i denote the number of columns of the form A^jb^i that were added to the basis by the algorithm. The numbers $\mu_1, \ldots, \mu_m$ are called the controllability indices of (A, B). Under what condition on the controllability indices is (A, B) controllable? (Controllability indices are related to the so-called Luenberger controllable canonical forms that generalize the CCF we've seen for SISO systems and which can be found by elementary row and column operations operating on the A and B matrices.)

   Solution: (A, B) is controllable if and only if $\mu_1 + \cdots + \mu_m = n$.

(c) According to the theory of Luenberger controllable canonical forms, any state space model with $n = 4, m = 2$ and controllability indices $\mu_1 = \mu_2 = 2$ can be put into the following form by a state space transformation for some values of the constants indicated:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & b & c & d \\ 0 & 0 & 0 & 1 \\ e & f & g & h \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & x \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

For what values of the constants are the controllablity indices for the above (A, B) given by $\mu_1 = \mu_2 = 2$? (This shows that not all values of the constants work.)

Solution: b_1 and Ab_1 are linearly independent. The next thing we need to hold is that $A^2 b^1 \in \text{span}[b\ Ab]$, or equivalently

$$\begin{bmatrix} b \\ a + b^2 + df \\ f \\ e + bf + fh \end{bmatrix} \in \mathcal{R} \begin{bmatrix} 0 & 1 \\ 1 & b \\ 0 & 0 \\ 0 & f \end{bmatrix} \tag{1}$$

From the third and fourth rows we see (1) holds if and only if $f = e = 0$. It then remains to check that appending columns b^2 and Ab^2 yields a full rank matrix. That matrix is

$$\begin{bmatrix} 0 & 1 & 0 & x \\ 1 & b & x & bx + d \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & h \end{bmatrix}$$

It has full rank (elementary column operations can make it upper triangular) so no more columns can be added by the algorithm. In summary, the given pair has controllability indices $\mu_1 = \mu_2 = 2$ if and only if $e = f = 0$.

2. **[Eigenvector criterion equivalent to PBH criterion]**
The Papov - Belevitch - Hautus criterion (aka Hautus Rosenbrock criterion) for controllability of (A, B) where A is $n \times n$ and B is $n \times m$ is that $\begin{bmatrix} (\lambda I - A) & B \end{bmatrix}$ have rank n (i.e. full rank) for all $\lambda \in \mathbb{C}$. (And the criterion for detectability is that the same hold for all $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$ – but this problem focuses on controllability.)

(a) Show that the PBH criterion is equivalent to the following *eigenvector criterion*: For every left eigenvector r^* of A it holds that $r^* B \neq \vartheta$. (By left eigenvector of A we mean $r^* A = \lambda r^*$ for some $\lambda \in \mathbb{C}$ or equivalently r is a right eigenvector of A^*.)

Solution: The PBH criterion doesn't hold if and only if there is $\lambda \in \mathbb{C}$ and a vector r so that $r^* \begin{bmatrix} (\lambda I - A) & B \end{bmatrix} = \vartheta$ which is equivalent to $\lambda r^* = r^* A$ and $r^* B = \vartheta$ which is equivalent to the eigenvector criterion not holding.

(b) Give a direct proof that if the eigenvector criterion does not hold then the controllability matrix $\mathcal{C}$ is not full rank.

Solution: If r^* is a left eigenvector of A and if $r^* B = \vartheta$ then $r^* A^i B = \lambda^i r^* B = \vartheta$ for $0 \leq i \leq n - 1$. That is, $r^* \mathcal{C} = \vartheta$ which implies that $\mathcal{C}$ is not full rank.

(c) Conversely we show that if the controllability matrix $\mathcal{C}$ is not full rank then the eigenvector criterion for controllability does not hold. The column span of $\mathcal{C}$ is Σ_c, the controllable subspace. (Some notation: $A^*\Sigma_c^\perp := \{A^*v : v \in \Sigma_c^\perp\}$.) (i) Show that $A^*\Sigma_c^\perp \subset \Sigma_c^\perp$ (i.e. $\Sigma_c^\perp$ is invariant under A^*). Therefore, if $\mathcal{C}$ is not full rank then $\Sigma_c^\perp$ has dimension one or more and A^* restricted to $\Sigma_c^\perp$ has at least one eigenvalue and at least one eigenvector r for that eigenvalue, because relative to any basis for $\Sigma_c^\perp$ the linear transformation is equivalent to multiplication by a square matrix. (ii) Show that the existence of such r implies that the eigenvector criterion does not hold.

Solution: (i) Combining the definitions of $\Sigma_c^\perp$ and $A^*\Sigma_c^\perp$ we have

$$\begin{aligned} A^*\Sigma_c^\perp &= \{A^*v : v^*A^kB = \vartheta \text{ for } 0 \le k \le n-1\} \\ &= \{A^*v : v^*A^kB = \vartheta \text{ for } 0 \le k \le n\} \quad \text{by Caley-Hamilton} \\ &\subset \{A^*v : v^*A^kB = \vartheta \text{ for } 1 \le k \le n\} \\ &= \{A^*v : (A^*v)^*A^{k-1}B = \vartheta \text{ for } 1 \le k \le n\} \subset \Sigma_c^\perp \end{aligned}$$

(ii) For such r, r^* is a left eigenvector of A and $r^*B = \vartheta$ because the columns of B are in Σ_c. Thus the eigenvector criterion does not hold.

3. **[Sorting out modes if all eigenvalues are distinct]**
Consider a standard LTI model with matrices A, B, C, D such that A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Let $M = [v^1 \ \cdots \ v^n]$ be the modal matrix formed by the corresponding eigenvectors and let $r^{1*}, \dots, r^{n*}$ be the corresponding dual basis, which are the rows of M^{-1} and are also left eigenvectors of A.

   (a) Write down the model $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ obtained using the state transformation $\bar{x} = M^{-1}x$. Carefully identify the rows, B^i, of $\bar{B}$ and the columns, γ^i, of $\bar{C}$ in terms of B, C and the eigenvectors.

   Solution: (This was done in class when controllability (Chapter 5) and observability (Chapter 6) were discussed.)

$$\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \bar{x} + \underbrace{\begin{bmatrix} B^1 \\ B^2 \\ \vdots \\ B^n \end{bmatrix}}_{\bar{B}} u$$

$$y = \underbrace{\begin{bmatrix} \gamma^1 & \gamma^2 & \cdots & \gamma^n \end{bmatrix}}_{\bar{C}} \bar{x} + Du$$

   where $B^i = r^{i*}B$, $\gamma^i = Cv^i$, and $\bar{D} = D$.

   (b) Under what conditions on the λ_i, B^i, γ_i is (A, B) stabilizable? Justify your answer directly, although you should also see that it is consistent with the PBH criterion for stabilizability.

   Solution: (A, B) is stabilizable if and only if $B^i \neq \vartheta$ for each i such that $Re(\lambda_i) \geq 0$. If there exists i such that $B^i = \vartheta$ and $\text{Re}(\lambda_i) \geq 0$ then the control has no effect on $\bar{x}_i(t)$ and $\bar{x}_i(t) = \bar{x}(0)e^{\lambda_i t} \nrightarrow 0$. If $B^i \neq \vartheta$ for all i such that $Re(\lambda^i) > 0$ then those coordinates form a controllable block because the functions $e^{\lambda_i t}$ are linearly independent. (Recall discussion of controllability.)

(c) Under what conditions on the λ_i, B^i, γ_i is (A, C) detectable? Justify your answer directly, although you should also see that it is consistent with the PBH criterion for detectability.

Solution: (A, C) is detectable if and only if $\gamma^i \neq \vartheta$ for each i such that $Re(\lambda_i) \geq 0$. If there exists i such the $\gamma^i = \vartheta$ and $\mathrm{Re}(\lambda_i) \geq 0$ then x_i has no effect on y and $\bar{x}_i(t) = \bar{x}(0)e^{\lambda_i t} \nrightarrow 0$. If $\gamma^i \neq \vartheta$ for all i such that $Re(\lambda^i) > 0$ then those coordinates form an observable block because the functions $e^{\lambda_i t}$ are linearly independent.

(d) Suppose the matrix $\bar{B}$ above is given by

$$\bar{B} = \begin{bmatrix} 0 & 79 & 0 \\ 1 & -14 & 0 \\ 0 & 0 & 43 \\ 0 & 0 & 1 \\ -19 & 0 & 19 \end{bmatrix}$$

What are the controllability indices of the original system? Is it controllable? (Hint: The indices are determined by which elements of $\bar{B}$ are nonzero.)

Solution: $(\mu_1, \mu_2, \mu_3) = (2, 1, 2)$, because two rows have a nonzero entry in column one. Of the remaining rows one has a nonzero entry in column two. And of the rows still remaining, two columns have a nonzero entry in the third column. Since the indices sum to n the original system is controllable.

4. **[Failure of static output feedback for the harmonic oscillator]**
Consider the harmonic oscillator with position measurements satisfying $\ddot{x} + x = u$, $y = x$.

(a) Show that it cannot be asymptotically stabilized (i.e. making $x(t) \to 0$) by static *output* feedback $u = -ky$ no matter what real value of k is chosen.

Solution: With $u = -ky$ the state satisfies $\ddot{x} = -(1 + k)x$ or in state space form with $x_1 = x$ and $x_2 = \dot{x}$,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(1+k) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues of A are $\pm j\sqrt{1+k}$. If $k \geq -1$ then both eigenvalues are on the imatinary axis and if $k < -1$ then one eigenvalue is strictly positive. Thus k cannot be selected to make the system asymptotically stable.

Note: In general the solution $y = x$ has the form $y(t) = ae^{j(\sqrt{1+k})t} + be^{-j(\sqrt{1+k})t}$ which can be expressed as $A\cos(\sqrt{1+k}\ t + \phi)$ if $k \geq -1$ and is the sum of two exponentials with real and opposite exponents if $k < -1$.

(b) Derive a static state feedback to stabilize the system and place the closed loop poles at -2 and -2.

Solution: The system model is given by:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Using state feedback $u = -Kx$ where $K = [k_1 \quad k_2]$ yields the closed loop system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 - k_1 & -k_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Setting the characteristic polynomial for A_{cl} equal to $(s+2)^2$ yields $s^2 + k_2 s + 1 + k_1 = s^2 + 4s + 4$ or $[k_1 \;\; k_2] = [3 \;\; 4]$.

(c) Derive a dynamic output feedback control that stabilizes the system using an observer and the certainty equivalence to state feedback, using the methodology of Chapter 7 of the notes. To be definite, if possible, make the poles of the observer -8 and -8 and the poles of the state feedback portion -2 and -2.

Solution: The joint model for the state and observer is given by:

$$\begin{aligned}\dot{x} &= Ax - BK\hat{x} \\ \dot{\hat{x}} &= (A - BK)\hat{x} + L(y - C\hat{x})\end{aligned}$$

By the separation principle we take $[k_1 \;\; k_2] = [3 \;\; 4]$ as in part (b) and then find $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ so that $A - LC$ has eigenvalues at -8, -8. The result is $L = \begin{bmatrix} 16 \\ 63 \end{bmatrix}$.

5. **[Amplitude and phase tracking via the internal model principle]**
Consider again the harmonic oscillator with position measurements satisfying $\ddot{x} + x = u$, $y = x$. Note that if $u = 0$ the oscillation will happen at the frequency 1 radian/sec (rad/s).

(a) Suppose we wish to apply a control law to double the oscillator frequency from its natural frequency of 1 rad/s to 2 rad/s. Find a *static* state feedback control law for doing so.

Solution: By inspection we see that using $u = -3x$ makes the closed loop differential equation $\ddot{x} = -4x$ which has general solution of the form $A\cos(2t + \phi)$. OR we could look at the second order state space model as in part (a) of the previous problem and see taking $k = 3$ makes the closed loop system eigenvalue $\pm j\sqrt{1+k} = \pm 2j$. Equivalently, we use $K = [k_1 \;\; k_2] = [3 \;\; 0]$ for the second order state space model.

(b) Suppose you wish not only to control the frequency to be 2 rad/s, but you wish to control the amplitude and phase of the output so that it tracks the reference signal $r(t) = 5\cos(2t)$. Using the internal model principle, devise a dynamic state feedback controller to achieve this objective. You should find a fourth order system model. Describe whether you can place the four poles arbitrarily and if so, describe how that would be done. (Hint: Since $(s^2+4)r = 0$ try using the control u such that $(s^2+4)u = v$, let $z = (s^2+4)x$ and find the fourth order system with states given by $z, e, \dot{e}$ where $e = y(t) - r(t)$, and control v. Then select a state feedback form for v, namely $v = [-k_1 \;\; -k_2]z - k_3 e - k_4 \dot{e}$ and translate this to describe the dynamic state feedback controller u.)

Solution: Let z, e, v be defined as in the hint. The state equations for the oringal system model are

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x\end{aligned}$$

We then find $\dot{z} = \dddot{x} + 4\dot{x} = A(\ddot{x} + 4x) + B(\ddot{u} + 4u) = Az + Bv$. Also $\ddot{e} = \ddot{y} - \ddot{r} = C\ddot{x} - \ddot{r} = C(z - 4x) + 4r = Cz - 4e$. This gives rise to the system state model

$$\frac{d}{dt}\begin{bmatrix} \dot{z} \\ e \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 1 \\ C & -4 & 0 \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} z \\ e \\ \dot{e} \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}}_{\tilde{B}} v$$

Expanding out the submatrices in $\tilde{A}$ and $\tilde{B}$ gives:

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -4 & 0 \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The controllability matrix for $(\tilde{A}, \tilde{B})$ is given by

$$\tilde{\mathcal{C}} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and is seen to be full rank, so state feedback can be used to place the four poles of this system arbitrarily. More specifically, the closed loop matrix is

$$\tilde{A}_{cl} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1-k_1 & -k_2 & -k_3 & -k_4 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -4 & 0 \end{bmatrix}$$

and $k_1, \ldots, k_4$ can be chosen to place the eigenvalues of $\tilde{A}_{cl}$ arbitrarily.
The realizable control law u is given by

$$\begin{aligned} u &= \frac{1}{1+s^2} v \\ &= \frac{1}{1+s^2} [-k_1 \quad k_2] z - k_3 e - k_4 \dot{e} \\ &= [-k_1 \quad -k_2] x + \left(\frac{-k_3 - k_4 s}{s^2+4} \right) e \end{aligned}$$

(c) The controller you found in part (b) was based on state feedback. Briefly describe how an observer can be introduced to derive an output feedback control law in an attempt to achieve the same objective as in part (b).

Solution: The idea is to replace x in the control law by $\hat{x}$ where $\dot{\hat{x}} = A\hat{x} + L(y - C\hat{x}) + Bu$. This would increase the order of the underlying system to six. By the separation principle we would select L to place the eigenvalues of $(A - LC)$ to have negative real parts with greater magnitude than those placed for the fourth order system above. Due to the underlying tracking in part (b) more analysis is required to see the performance of this observer in this context. NOTE: Another idea would be to set k_2 above equal to zero and see how well the poles of $\tilde{A}_{cl}$ can be placed by selection of the other k's. With $k_2 = 0$ the controller from part (b) requires only output feedback.