University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 8
Feedback: Tracking and Disturbance Rejection

Due: Wednesday, October 30, 11:59pm
Reading: Course notes, Chapter 8 (also review Chapter 7)

1. **[Controllability indices]**
   Consider the controllability matrix $\mathcal{C} = [B \ AB \ A^2B \ \cdots \ A^{n-1}B]$ for a linear system model, where A is $n \times n$ and B is $n \times m$ for positive integers m, n. Let $b^1, \ldots, b^m$ denote the columns of B. Let $\bar{\mathcal{C}}$ be obtained by reordering the columns of $\mathcal{C}$ as follows (such reordering does not change the column span):

$$\overline{\mathcal{C}} = \left[b^1 \ Ab^1 \ A^2b^1 \ \cdots \ A^{n-1}b^1 \ b^2 \ Ab^2 \ \cdots \ A^{n-1}b^2 \ b^3 \ \ \cdots \ A^{n-1}b^m\right].$$

   A basis for the column span of $\mathcal{C}$, or equivalently, the range space of $\mathcal{C}$, can be found by the following algorithm. Consider the columns of $\overline{\mathcal{C}}$ one by one from left to right, and add any column to the basis that is not in the span of the columns before it.

   (a) Show that whenever a column of the form $A^j b^i$ is not included in the basis then any column of the form $A^{j'} b^i$ with $j' > j$ will not be included. (Hint: Start by considering the columns with $i = 1$.)

   (b) Let μ_i denote the number of columns of the form $A^j b^i$ that were added to the basis by the algorithm. The numbers $\mu_1, \ldots, \mu_m$ are called the controllability indices of (A, B). Under what condition on the controllability indices is (A, B) controllable? (Controllability indices are related to the so-called Luenberger controllable canonical forms that generalize the CCF we've seen for SISO systems and which can be found by elementary row and column operations operating on the A and B matrices.)

   (c) According to the theory of Luenberger controllable canonical forms, any state space model with $n = 4, m = 2$ and controllability indices $\mu_1 = \mu_2 = 2$ can be put into the following form by a state space transformation for some values of the constants indicated:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & b & c & d \\ 0 & 0 & 0 & 1 \\ e & f & g & h \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & x \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

   For what values of the constants are the controllablity indices for the above (A, B) given by $\mu_1 = \mu_2 = 2$? (This shows that not all values of the constants work.)

2. **[Eigenvector criterion equivalent to PBH criterion]**
   The Papov - Belevitch - Hautus criterion (aka Hautus Rosenbrock criterion) for controllability of (A, B) where A is $n \times n$ and B is $n \times m$ is that $\begin{bmatrix} (\lambda I - A) & B \end{bmatrix}$ have rank n (i.e. full rank) for all $\lambda \in \mathbb{C}$. (And the criterion for detectability is that the same hold for all $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$ – but this problem focuses on controllability.)

   (a) Show that the PBH criterion is equivalent to the following *eigenvector criterion*: For every left eigenvector r^* of A it holds that $r^*B \neq \vartheta$. (By left eigenvector of A we mean $r^*A = \lambda r^*$ for some $\lambda \in \mathbb{C}$ or equivalently r is a right eigenvector of A^*.)

(b) Give a direct proof that if the eigenvector criterion does not hold then the controllability matrix $\mathcal{C}$ is not full rank.

(c) Conversely we show that if the controllability matrix $\mathcal{C}$ is not full rank then the eigenvector criterion for controllability does not hold. The column span of $\mathcal{C}$ is Σ_c, the controllable subspace. (Some notation: $A^*\Sigma_c^\perp := \{A^*v : v \in \Sigma_c^\perp\}$.) (i) Show that $A^*\Sigma_c^\perp \subset \Sigma_c^\perp$ (i.e. $\Sigma_c^\perp$ is invariant under A^*). Therefore, if $\mathcal{C}$ is not full rank then $\Sigma_c^\perp$ has dimension one or more and A^* restricted to $\Sigma_c^\perp$ has at least one eigenvalue and at least one eigenvector r for that eigenvalue, because relative to any basis for $\Sigma_c^\perp$ the linear transformation is equivalent to multiplication by a square matrix. (ii) Show that the existence of such r implies that the eigenvector criterion does not hold.

3. **[Sorting out modes if all eigenvalues are distinct]**
Consider a standard LTI model with matrices A, B, C, D such that A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Let $M = [v^1 \ \cdots \ v^n]$ be the modal matrix formed by the corresponding eigenvectors and let $r^{1*}, \dots, r^{n*}$ be the corresponding dual basis, which are the rows of M^{-1} and are also left eigenvectors of A.

(a) Write down the model $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ obtained using the state transformation $\bar{x} = M^{-1}x$. Carefully identify the rows, B^i, of $\bar{B}$ and the columns, γ^i, of $\bar{C}$ in terms of B, C and the eigenvectors.

(b) Under what conditions on the λ_i, B^i, γ_i is (A, B) stabilizable? Justify your answer directly, although you should also see that it is consistent with the PBH criterion for stabilizability.

(c) Under what conditions on the λ_i, B^i, γ_i is (A, C) detectable? Justify your answer directly, although you should also see that it is consistent with the PBH criterion for detectability.

(d) Suppose the matrix $\bar{B}$ above is given by

$$\bar{B} = \begin{bmatrix} 0 & 79 & 0 \\ 1 & -14 & 0 \\ 0 & 0 & 43 \\ 0 & 0 & 1 \\ -19 & 0 & 19 \end{bmatrix}$$

What are the controllability indices of the original system? Is it controllable? (Hint: The indices are determined by which elements of $\bar{B}$ are nonzero.)

4. **[Failure of static output feedback for the harmonic oscillator]**
Consider the harmonic oscillator with position measurements satisfying $\ddot{x} + x = u$, $y = x$.

(a) Show that it cannot be asymptotically stabilized (i.e. making $x(t) \to 0$) by static *output* feedback $u = -ky$ no matter what real value of k is chosen.

(b) Derive a static state feedback to stabilize the system and place the closed loop poles at -2 and -2.

(c) Derive a dynamic output feedback control that stabilizes the system using an observer and the certainty equivalence to state feedback, using the methodology of Chapter 7 of the notes. To be definite, if possible, make the poles of the observer -8 and -8 and the poles of the state feedback portion -2 and -2.

5. [**Amplitude and phase tracking via the internal model principle**]
Consider again the harmonic oscillator with position measurements satisfying $\ddot{x} + x = u$, $y = x$. Note that if $u = 0$ the oscillation will happen at the frequency 1 radian/sec (rad/s).

   (a) Suppose we wish to apply a control law to double the oscillator frequency from its natural frequency of 1 rad/s to 2 rad/s. Find a *static* state feedback control law for doing so.

   (b) Suppose you wish not only to control the frequency to be 2 rad/s, but you wish to control the amplitude and phase of the output so that it tracks the reference signal $r(t) = 5\cos(2t)$. Using the internal model principle, devise a dynamic state feedback controller to achieve this objective. You should find a fourth order system model. Describe whether you can place the four poles arbitrarily and if so, describe how that would be done. (Hint: Since $(s^2 + 4)r = 0$ try using the control u such that $(s^2 + 4)u = v$, let $z = (s^2 + 4)x$ and find the fourth order system with states given by $z, e, \dot{e}$ where $e = y(t) - r(t)$, and control v. Then select a state feedback form for v, namely $v = [-k_1 \quad -k_2]z - k_3 e - k_4 \dot{e}$ and translate this to describe the dynamic state feedback controller u.)

   (c) The controller you found in part (b) was based on state feedback. Briefly describe how an observer can be introduced to derive an output feedback control law in an attempt to achieve the same objective as in part (b).