University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 7: Problems and Solutions
Feedback: Pole Placement

Due: Wednesday, October 23, 11:59pm
Reading: Course notes, Chapter 7 (also review Chapter 6)

1. **[Duality and state transformation]**
Consider LTI system I with matrices A_I, B_I, C_I, D_I with the usual dimensions. Let $\bar{A}_I, \bar{B}_I, \bar{C}_I, \bar{D}_I$ denote the system obtained from system I by state transformation $\bar{x}_I = Px_I$. Show that the duals of these two systems are related to each other by state transformation, and express the state transformation matrix for the dual systems in terms of P. Use logical notation in your answer with system II being the dual of system I.

   Solution: On one hand, by definition, the dual of system I is system II with matrices
$$(A_{II}, B_{II}, C_{II}, D_{II}) = (-A_I^*, C_I^*, B_I^*, D_I^*). \tag{1}$$
On the other hand, system I after state transformation is given by
$$(\bar{A}_I, \bar{B}_I, \bar{C}_I, \bar{D}_I) = (PA_IP^{-1}, PB_I, C_IP^{-1}, D_I),$$
and thus the dual of (system I after state transformation) is given by:
$$(-\bar{A}_I^*, \bar{C}_I^*, \bar{B}_I^*, \bar{D}_I^*) = (-P^{-1*}A_I^*P^*, P^{-1*}C_I^*, B_I^*P^*, D_I^*). \tag{2}$$
(In the above P^{-1*} stands for $(P^{-1})^*$ which is also equal to $(P^*)^{-1}$, because $I = (PP^{-1})^* = (P^{-1})^*P^*$.) Observe that the righthand side of (2) is the result of applying the state transformation matrix P^{-1*} to the righthand side of (1). Therefore, system II is transformed to the dual of (system I after state transformation) by the state transformation matrix P^{-1*}.

   Symbolically, we can summarize this answer by the following diagram:
$$\begin{array}{ccc} \text{system I} & \xrightarrow{P} & \text{system } \bar{\text{I}} \\ \updownarrow \text{duals} & & \updownarrow \text{duals} \\ \text{system II} & \xrightarrow{P^{-1*}} & \text{system } \overline{\text{II}} \end{array}$$

2. **[Kalman Observability Canonical Form (KOCF)]**
Consider an LTI system with matrices A, B, C, D with the usual dimensions. The last sentence of Section 6.4 of the course notes states that the KOCF can be found from the KCCF by duality. The state transformation needed to get the KOCF can also be found by duality–see part (c) below. But first, we focus on directly finding the KOCF. (Recall from the course notes and lectures that to get the KCCF we can select the first columns of P^{-1} to be a basis for the column span of the controllability matrix $\mathcal{C}$.) Let $\text{rank}(\mathcal{O}) = n_1$.

   (a) Select P so that its first n_1 rows form a basis for the row span of $\mathcal{O}$. (So if x is in the unobservable subspace, its first n_1 coordinates after state transformation will be zero.) Show that the system obtained from A, B, C, D by the state space transformation $\bar{x} = Px$ has the KOCF. (Hint: $PP^{-1} = I$. Using the notation from the notes, it must be shown that the system after state transformation satisfies: the upper right $n_1 \times (n-n_1)$ block of $\bar{A}$ is zero, the last $n-n_1$ columns of $\bar{C}$ are zero, and (A_o, C_o) is observable.)

Solution: Suppose P is a matrix as described. Since $PP^{-1} = I$, the last $n-n_1$ columns of P^{-1} are orthogonal to the first n_1 rows of P. That is, they are in the null space of $\mathcal{O}$. (Moreover, since there are $n-n_1$ of them and they are linearly independent – being part of P^{-1} – they form a basis for the null space of $\mathcal{O}$.) By the form of $\mathcal{O}$ and the Caley-Hamilton theorem, for any vector v in the null space of $\mathcal{O}$, Av is also in the null space of $\mathcal{O}$. So the last $n-n_1$ columns of AP^{-1} are in the null space of $\mathcal{O}$. It follows by the choice of P that the upper right $n_1 \times (n-n_1)$ block of $\bar{A} = PAP^{-1}$ is the zero matrix. Similarly, since the rows of C are in the row span of $\mathcal{O}$, the last $n-n_1$ rows of CP^{-1} are zero vectors.

It remains to show that (A_o, C_o) is observable. Since state transformation does not change the dimension of the observable subspace,

$$n_1 = \mathrm{rank}(\mathcal{O}) = \mathrm{rank}(\bar{\mathcal{O}}) = \mathrm{rank}\begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}^{n-1}\bar{A} \end{bmatrix}$$

$$= \mathrm{rank}\begin{bmatrix} C_o & 0 \\ C_oA_o & 0 \\ \vdots & \vdots \\ \bar{C}_oA_o^{n-1} & 0 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} \bar{C}_o \\ \bar{C}_o\bar{A}_o \\ \vdots \\ \bar{C}_o^{\,k-1}\bar{A}_o \end{bmatrix}$$

Therefore (A_o, C_o) is observable.

(b) Show that another way to specify a suitable matrix P is to let the last $n-n_1$ columns of P^{-1} be a basis for the null space of $\mathcal{O}$ and then selecting the first n_1 columns of P^{-1} to make P nonsingular. (Hint: There is a short argument relying on part (a).)

Solution: Again using $PP^{-1} = I$ we see the first n_1 rows of P must be in the row span of $\mathcal{O}$ (they are orthogonal to the orthogonal complement of the row span). Since they are linearly independent they must be a basis for the row span of P. So P falls within the construction of part (a) so the KOCF follows.

(c) Explain how to use the previous problem on **Duality and state transformation** to rederive the state transformation for observability described in part (a) of this problem. Specifically, let P_c denote the state transformation matrix for putting the dual of system (A, B, C, D) into KCCF form and express P from part (a) in terms of P_c.

Solution: The controllability matrix for the dual of the original system (leaving out the ± 1's which don't change the column span) is $\mathcal{C}_{\text{dual}} = (C^* \;\; A^*C^* \;\; \cdots \;\; (A^*)^{n-1}C^*)$ and, as recalled in the problem statement, we select the first n_1 columns of P_c^{-1} to be a basis for the column span of $\mathcal{C}_{\text{dual}}$. That ensures the state transformation P_c^{-1} maps the dual to the original system to a system in KCCF form. Equivalently, we select P_c^{-1*} so that its first n_1 rows form a basis for the row span of $\mathcal{C}^*_{\text{dual}} = \mathcal{O}$. By the previous problem on **Duality and state transformation**, P_c^{-1*} is also the corresponding state transformation matrix P for the original system, which transforms the original system to one in KOCF form, the dual of KCCF form. We thus have $P = P_c^{-1*}$.

To summarize, one way to find the state transformation matrix P that brings the original system to KOCF form is to find the state transformation matrix P_c that brings the dual of the original system to KCCF form and then let $P = P_c^{-1*}$.

3. **[Output feedback stabilization example]**
Consider the LTI state space model

$$\dot{x} = \begin{bmatrix} -1 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = [1\ 0\ 1]x$$

(a) Find a KCCF form for the system by finding a suitable state transformation. Is the original system stabilizable?

Solution: The controllability matrix, $\mathcal{C} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}$, has rank two and its column span is equal to span $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$. A reasonable choice of state transformation matrix is $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. We find $P^{-1} = P$ and the corresponding KCCF is given by

$$\dot{x} = \left[\begin{array}{cc|c} -1 & 3 & 3 \\ 0 & 2 & 0 \\ \hline 0 & 0 & -1 \end{array}\right] x + \left[\begin{array}{c} 1 \\ 1 \\ \hline 0 \end{array}\right] u$$
$$y = [1\ \ 1\ \ 0]x$$

(The lines within the matrices deliniate the subsystems but aren't required for your answer.) Yes, the original system is stabilizable because the 1×1 submatrix for the uncontrollable part of x is [-1], which is Hurwitz.

NOTE: The KCCF depends on the choice of P and is not unique. However, the dimensions and eigenvalues of the submatrices A_c and $A_{\bar{c}}$ are unique and all correct choices of P will lead to the same conclusion that the original system is stabilizable.

(b) Find a KOCF form for the system by finding a suitable state transformation. Is the original system detectable?

Solution: The observability matrix, $\mathcal{O} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 2 \\ 1 & -6 & 4 \end{bmatrix}$, has full rank, so the original system is observable and it is therefore already in KOCF form. Yes, the original system is detectable because it is observable.

(c) Suppose output feedback is used in an attempt to stabilize the overall system using feedback $u = K\hat{x}$, where $\dot{\hat{x}} = (A - BK)\hat{x} + Bu + L(y - C\hat{x})$ represents an observer. The system has six eigenvalues (counting multiplicity). Suppose we try to find L and K so that the eigenvalues of the observer are $-6, -6, -6$ and the remaining eigenvalues of the state are $-2, -2, -2$. Is that possible? If not, what would be a way to get close to that? (You don't need to find the K and L matrices.)

Solution: The controllable subspace has dimension 2 and by selecting a feedback matrix K we can select two of the eigenvalues arbitrarily while the third eigenvalue is stuck at -1. Since the system is observable we can select L to make the dynamics of the observer have whatever eigenvalues we want. Thus, we cannot make the eigenvalues of the closed loop observer feedback system $-2, -2, -2, -6, -6, -6$ as desired, but we could, for example, make the eigenvalues $-2, -2, -1, -6, -6, -6$. Note that we are implicitly relying on the separation principle to justify focusing separately on pole placement for the control subsystem and the observer.

4. **[Minimal realizations and effects of feedback on controllability and observability]** Determine whether each of the following statements is true or false for an LTI system A, B, C, D with the usual notation and justify each answer with either a proof or counter example.

(a) Any two minimal realizations of a SISO transfer function $P(s)$ are related to each other by a state transformation.

Solution: True. Write $P(s) = \frac{s^{n-1}\beta_1 + \cdots + \beta_n}{s^n + s^{n-1}\alpha_1 + \cdots + \alpha_n} + d = \frac{N(s)}{D(s)} + d$ where the polynomials N and D have no common roots. The CCF realization of $P(s)$ has order n and no other realization can have a smaller order, so any minimal realization of $P(s)$ has order n and the characteristic polynomial of its A matrix, $\triangle(s) := \det(sI - A)$, must equal $D(s)$. Also, any minimal realization of $P(s)$ must be controllable. Hence, by the problem on **controllable canonical form** in Problem set 5, any minimal realization of $P(s)$ must be equivalent up to state transformation to the CCF realization. Since equivalence up to state transformation is transitive, any two minimal realizations of P must be equivalent to each other.

(b) State feedback does not change the controllable subspace. In other words, the controllable subspaces of (A, B) and $(A - BK, B)$ are identical.

Solution: True. The controllable subspaces are the column spans of the controllability matrices. The controllability matrices of the open loop and closed loop systems are:

$$\mathcal{C} = (B, AB, A^2B, \ldots, A^{n-1}B)$$
$$\mathcal{C}_{\text{cl}} = (B, (A - BK)B, (A - BK)^2B, \ldots, (A - BK)^{n-1}B)$$

By an induction argument on j we can show that the column spans of the first j blocks of $\mathcal{C}$ and $\mathcal{C}_{\text{cl}}$ are the same for $1 \le j \le n-1$. The key observation is that $(A-BK)^jB = [A^jB$ plus terms of the form A^iBZ with $0 \le i \le j-1.]$

(c) State feedback does not change the unobservable subspace. In other words, the unobservable subspaces of (A, C) and $(A - BK, C)$ are identical.

Solution: False. For example, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad K = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

then the original system is observable (the unobservable subspace is $\{\vartheta\}$ with dimension zero) whereas the closed loop system has $A_{\text{cl}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and a one dimensional unobservable subspace, $\Sigma_{\bar{o}}^{\text{cl}} = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(d) Static output feedback does not change the controllable subspace. In other words, the controllable subspaces of (A, B) and $(A - BKC, B)$ are identical.

Solution: True by part (b) – output feedback is a special case of state feedback.

(e) Static output feedback does not change the unobservable subspace. In other words, the unobservable subspaces of (A, C) and $(A - BKC, C)$ are identical.

Solution: True. In fact, the row spans of the observability matrices of (A, C) and $(A - LC, C)$ are the same for any L, including $L = BK$. This follows from part (b) and duality. (If you don't use duality you will see a story similar to part (b) with the observability matrices.)

5. **[Pole placement to cancel a zero]**
Consider the transfer function $P(s) = \frac{s+22}{(s+18)(s+20)}$.

(a) Find the CCF realization of $P(s)$ and then find a matrix K_c so that state feedback $u = r - Kx$ (where r is the input to the closed loop system) places the poles at -18 and -22. What is the closed loop transfer function? What happened to the zero at -22? Is the closed loop system controllable? Is it observable?

Solution: Since $P(s) = \frac{s+22}{s^2+38s+360}$, the CCF is given by

$$A = \begin{bmatrix} 0 & 1 \\ -360 & -38 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 22 & 1 \end{bmatrix}$$

We aim for the denominator of the closed loop transfer function to be $(s+18)(s+22) = s^2 + 40s + 396$. With $K_c = [k_1 \;\; k_2]$ we find $A - BK_c = \begin{bmatrix} 0 & 1 \\ -360 - k_1 & -38 - k_2 \end{bmatrix}$ so to get the CCF for the new denominator we set $K_c = [36, 2]$ so that $A_{cl} = A - BK_c = \begin{bmatrix} 0 & 1 \\ -396 & -40 \end{bmatrix}$. The closed loop tranfer function is then

$$P(s) = C(Is - A_{cl})^{-1}B = \frac{s+22}{(s+18)(s+22)} = \frac{1}{s+18} \tag{3}$$

The pole placed at 22 cancelled the zero that was at 22. The closed loop system is not minimal so it can't be both controllable and observable. Checking the controllability matrix $\mathcal{C}$ and observability matrix $\mathcal{O}$ we find that the closed loop system is controllable but not observable. (That also follows from the fact that state feedback does not change whether a system is controllable.)

(b) Repeat part (b) but this time first find the modal realization of P. Then continue as in part (a) to find a matrix K_m so that state feedback $u = r - K_m x$ places the closed loop poles to -18 and -22. Again find the closed loop transfer function and see what happened to the zero at -22.

Solution: By partial fraction expansion, $P(s) = \frac{2}{s+18} - \frac{1}{s+20}$, so a modal realization is

$$A = \begin{bmatrix} -18 & 0 \\ 0 & -20 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 2 & -1 \end{bmatrix}.$$

Setting $K_m = [k_1 \;\; k_1]$ and equating $\det(A - BK_m) = \det \begin{bmatrix} -18 - k_1 & 0 - k_2 \\ -k_1 & -20 - k_2 \end{bmatrix}$ to $s^2 + 40s + 396$ we find $K_m = [0 \;\; 2]$. This gives the closed loop matrix $A_{cl} = A - BK_m =$

$\begin{bmatrix} -18 & -2 \\ 0 & -22 \end{bmatrix}$. We find that the closed loop transfer function is given by (3)–it is the same as in part (a). As in part (a) we find the closed loop system to be controllable but not observable.

(c) Find a nonsingular 2×2 state transformation matrix P to show that the open loop realizations (i.e. before the feedback was found) in parts (a) and (b) are the same up to state transformation, with $\bar{x} = Px$, where x is the state for your solution to part (a) and $\bar{x}$ is the state for your solution to part (b). Are the two closed loop systems similarly related using the same P?

Solution: Letting $\mathcal{C}_c$ denote the controllability matrix for the open loop CCF realization found in part (a) and $\mathcal{C}_m$ denote the controllability matrix for the open loop modal realization found in part (b), we find the state transformation matrix for mapping from the modal realization to the CCF is $P = \mathcal{C}_c \mathcal{C}_m^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -9 & 10 \end{bmatrix}$. Equivalently, the state transformation matrix for mapping from the CCF realization to the modal realization is $P^{-1} = \begin{bmatrix} 20 & 1 \\ 18 & 1 \end{bmatrix}$. The respective feedback matrices are related by the same state transformation: $K_m = K_c P$ and the two closed loop systems are the same up to the state transformation again given by P.

NOTE: The B and C matrices in the modal representation in part (b) are not unique. For example B and C^T could be swapped, and that would give modal to CCF transformation matrix $\begin{bmatrix} 1/4 & 1/2 \\ -9/2 & -10 \end{bmatrix}$ and CCF to modal transformation matrix $\begin{bmatrix} 40 & 2 \\ -18 & -1 \end{bmatrix}$.