University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 7
Feedback: Pole Placement

Due: Wednesday, October 23, 11:59pm
Reading: Course notes, Chapter 7 (also review Chapter 6)

1. [**Duality and state transformation**]
Consider LTI system I with matrices A_I, B_I, C_I, D_I with the usual dimensions. Let $\bar{A}_I, \bar{B}_I, \bar{C}_I, \bar{D}_I$ denote the system obtained from system I by state transformation $\bar{x}_I = Px_I$. Show that the duals of these two systems are related to each other by state transformation, and express the state transformation matrix for the dual systems in terms of P. Use logical notation in your answer with system II being the dual of system I.

2. [**Kalman Observability Canonical Form (KOCF)**]
Consider an LTI system with matrices A, B, C, D with the usual dimensions. The last sentence of Section 6.4 of the course notes states that the KOCF can be found from the KCCF by duality. The state transformation needed to get the KOCF can also be found by duality–see part (c) below. But first, we focus on directly finding the KOCF. (Recall from the course notes and lectures that to get the KCCF we can select the first columns of P^{-1} to be a basis for the column span of the controllability matrix $\mathcal{C}$.) Let $\text{rank}(\mathcal{O}) = n_1$.

   (a) Select P so that its first n_1 rows form a basis for the row span of $\mathcal{O}$. (So if x is in the unobservable subspace, its first n_1 coordinates after state transformation will be zero.) Show that the system obtained from A, B, C, D by the state space transformation $\bar{x} = Px$ has the KOCF. (Hint: $PP^{-1} = I$. Using the notation from the notes, it must be shown that the system after state transformation satisfies: the upper right $n_1 \times (n - n_1)$ block of $\bar{A}$ is zero, the last $n - n_1$ columns of $\bar{C}$ are zero, and (A_o, C_o) is observable.)

   (b) Show that another way to specify a suitable matrix P is to let the last $n - n_1$ columns of P^{-1} be a basis for the null space of $\mathcal{O}$ and then selecting the first n_1 columns of P^{-1} to make P nonsingular. (Hint: There is a short argument relying on part (a).)

   (c) Explain how to use the previous problem on **Duality and state transformation** to rederive the state transformation for observability described in part (a) of this problem. Specifically, let P_c denote the state transformation matrix for putting the dual of system (A, B, C, D) into KCCF form and express P from part (a) in terms of P_c.

3. [**Output feedback stabilization example**]
Consider the LTI state space model

$$\dot{x} = \begin{bmatrix} -1 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = [1\ 0\ 1]x$$

   (a) Find a KCCF form for the system by finding a suitable state transformation. Is the original system stabilizable?

(b) Find a KOCF form for the system by finding a suitable state transformation. Is the original system detectable?

(c) Suppose output feedback is used in an attempt to stabilize the overall system using feedback $u = K\hat{x}$, where $\dot{\hat{x}} = (A - BK)\hat{x} + Bu + L(y - C\hat{x})$ represents an observer. The system has six eigenvalues (counting multiplicity). Suppose we try to find L and K so that the eigenvalues of the observer are $-6, -6, -6$ and the remaining eigenvalues of the state are $-2, -2, -2$. Is that possible? If not, what would be a way to get close to that? (You don't need to find the K and L matrices.)

4. **[Minimal realizations and effects of feedback on controllability and observability]** Determine whether each of the following statements is true or false for an LTI system A, B, C, D with the usual notation and justify each answer with either a proof or counter example.

(a) Any two minimal realizations of a SISO transfer function $P(s)$ are related to each other by a state transformation.

(b) State feedback does not change the controllable subspace. In other words, the controllable subspaces of (A, B) and $(A - BK, B)$ are identical.

(c) State feedback does not change the unobservable subspace. In other words, the unobservable subspaces of (A, C) and $(A - BK, C)$ are identical.

(d) Static output feedback does not change the controllable subspace. In other words, the controllable subspaces of (A, B) and $(A - BKC, B)$ are identical.

(e) Static output feedback does not change the unobservable subspace. In other words, the unobservable subspaces of (A, C) and $(A - BKC, C)$ are identical.

5. **[Pole placement to cancel a zero]** Consider the transfer function $P(s) = \frac{s+22}{(s+18)(s+20)}$.

(a) Find the CCF realization of $P(s)$ and then find a matrix K_c so that state feedback $u = r - Kx$ (where r is the input to the closed loop system) places the poles at -18 and -22. What is the closed loop transfer function? What happened to the zero at -22? Is the closed loop system controllable? Is it observable?

(b) Repeat part (b) but this time first find the modal realization of P. Then continue as in part (a) to find a matrix K_m so that state feedback $u = r - K_m x$ places the closed loop poles to -18 and -22. Again find the closed loop transfer function and see what happened to the zero at -22.

(c) Find a nonsingular 2×2 state transformation matrix P to show that the open loop realizations (i.e. before the feedback was found) in parts (a) and (b) are the same up to state transformation, with $\bar{x} = Px$, where x is the state for your solution to part (a) and $\bar{x}$ is the state for your solution to part (b). Are the two closed loop systems similarly related using the same P?