ECE 515/ME 540: Problem Set 6: Problems and Solutions
Observability, Duality, and Minimality

Due: Wednesday, October 16, 11:59pm
Reading: Course notes, Chapter 6

1. **[Adjoint linear system]**
Let ϕ denote the state transition matrix for the LTV system $\dot{x} = A(t)x$ in $\mathbb{R}^n$ and let ϕ_a denote the state transition matrix for the associated *adjoint system* defined by $\dot{z} = -A^*(t)z$ using the same time varying A.)

   (a) Find an expression for $\frac{d}{dt}\phi(t_0, t)$ by using the fact $\frac{d}{dt}\{\phi(t_0, t)\phi(t, t_0)\} = 0_{n\times n}$ and removing the common factor $\phi(t, t_0)$.
   Solution: By the product rule for differentiation,
$$\begin{aligned} 0_{n\times n} = \frac{d}{dt}\{\phi(t_0,t)\phi(t,t_0)\} &= \left\{\frac{d}{dt}\phi(t_0,t)\right\}\phi(t,t_0) + \phi(t_0,t)\frac{d}{dt}\phi(t,t_0) \\ &= \left\{\frac{d}{dt}\phi(t_0,t)\right\}\phi(t,t_0) + \phi(t_0,t)A(t)\phi(t,t_0) \\ &= \left\{\frac{d}{dt}\phi(t_0,t) + \phi(t_0,t)A(t)\right\}\phi(t,t_0) \end{aligned}$$
   Since the matrix $\phi(t, t_0)$ is nonsingular it follows that $\frac{d}{dt}\phi(t_0, t) = -\phi(t_0, t)A(t)$.
   ALTERNATIVE: Another correct solution would be to not use the differential equation for ϕ. From the first line above we could deduce:
$$\frac{d}{dt}\phi(t_0,t) = -\phi(t_0,t)\left\{\frac{d}{dt}\phi(t,t_0)\right\}\phi^{-1}(t,t_0).$$
   With this answer to part (a), we can still do part (b). First, if we use $\frac{d}{dt}\phi(t, t_0) = A(t)\phi(t, t_0)$ we get $\frac{d}{dt}\phi(t_0, t) = -\phi(t_0, t)A(t)$ and proceed for the rest of part (b) as below.

   (b) Show that $\phi_a(t, t_0) = \phi^*(t_0, t)$. (Hint: For t_0 fixed, $\phi(t, t_0)$ is determined for all t by $\frac{d}{dt}\phi(t, t_0) = A(t)\phi(t, t_0)$ along with $\phi(t_0, t_0) = I$. Similarly, $\phi_a(t, t_0)$ is determined by $\frac{d}{dt}\phi_a(t, t_0) = -A^*(t)\phi_a(t, t_0)$ along with $\phi_a(t_0, t_0) = I$.)
   Solution: By part (a),
$$\frac{d}{dt}\phi^*(t_0,t) = -(\phi(t_0,t)A(t))^* = -A^*(t)\phi^*(t_o,t).$$
   and of course $\phi^*(t_0, t_0) = I$. Thus $\phi^*(t_0, t)$ satisfies the differential equation that determines $\phi_a(t, t_0)$, so they are equal.

   (c) Show that $z^*(t)x(t) = z^*(0)x(0)$ for all t in two different ways: (i) by differentiation and (ii) using the state transition matrices.
   Solution: (i)
$$\frac{d}{dt}\{z^*(t)x(t)\} = (-A^*(t)z(t))^*x(t) + z^*(t)A(t)x(t) = z^*(t)(-A(t)+A(t))x(t) = \theta.$$
   (ii) $z^*(t)x(t) = (\phi_a(t, 0)z(0))^*\phi(t, 0)x(0) = z^*(0)\phi(0, t)\phi(t, 0)x(0) = z^*(0)x(0)$.

2. **[Classification of first order LTI systems]**
Consider the first-order SISO LTI system model

$$\begin{aligned}\dot{x} &= ax + bu \\ y &= cx + du.\end{aligned}$$

(a) Under what conditions on a, b, c, d is the system controllable?
Solution: Controllable if and only if $b \neq 0$ because, for example, the controllability matrix is the 1×1 matrix $\mathcal{C} = [b]$.

(b) Under what conditions on a, b, c, d is the system observable?
Solution: Observable if and only if $c \neq 0$ because, for example, the observability matrix is the 1×1 matrix $\mathcal{O} = [c]$.

(c) Under what conditions on a, b, c, d is the system internally asymptotically stable (i.e. if there is zero control the state converges to zero from any initial condition)?
Solution: Asymptotically stable if and only if $Re(a) < 0$. In other words the A matrix is Hurwitz. The 1×1 matrix $[a]$ has a as its only eigenvalue.

(d) Under what condition on a, b, c, d is the system a minimal realization?
Solution: Realization in minimal if and only if $b \neq 0$ and $c \neq 0$. Same as being both controllable and observable.

3. **[Some realizations that are not minimal]**
Consider the transfer function

$$P(s) = \frac{s+2}{(s+2)(s+5)} = \frac{1}{s+5}.$$

(a) Obtain the second order state space realization in controllable canonical form (CCF). Is it controllable? Is it observable?
Solution: Since $P(s) = \frac{s+2}{s^2+7s+10}$, the CCF is given by

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 2 & 1 \end{bmatrix} x\end{aligned}$$

It is controllable (CCF is always controllable or we see $\mathcal{C} = \begin{bmatrix} 0 & 1 \\ 1 & -7 \end{bmatrix}$. is full rank)

The controllability matrix is given by $\mathcal{O} = \begin{bmatrix} 2 & 1 \\ -10 & -5 \end{bmatrix}$. It does not have full rank so the system is not observable. We could have deduced that from the fact the second order system is not minimal because $P(s) = \frac{1}{s+5}$ which implies there is a first order state space realization.

(b) Obtain the second order state space realization in observable canonical form (OCF). Is it controllable? Is it observable?
Solution: The OCF is given by

$$\begin{aligned}\dot{x} &= \begin{bmatrix} -7 & 1 \\ -10 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x\end{aligned}$$

It is observable (OCF is always observable or we see $\mathcal{O} = \begin{bmatrix} 1 & 0 \\ -7 & 1 \end{bmatrix}$ is full rank). The controllability matrix is given by $\mathcal{O} = \begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix}$. It does not have full rank so the system is not controllable, as we could also deduce from the non-minimality of the realization.

(c) Obtain the second order state space realization in modal form. Is it controllable? Is it observable?

Solution: Since $P(s) = \frac{1}{s+5} + \frac{0}{s+2}$, the second order modal realization is:

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

Checking $\mathcal{C} = \begin{bmatrix} 0 & 0 \\ 1 & -5 \end{bmatrix}$ and $\mathcal{O} = \begin{bmatrix} 1 & 1 \\ -2 & -5 \end{bmatrix}$ shows this realization is not controllable but it is observable.

4. **[Kalman controllability decomposition]**
Consider the LTI state space system $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & -3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

Find a nonsingular matrix P so that the corresponding change of coordinates brings the system into the Kalman controllability canonical form as in Chapter 5. This is not the same as the controllable canonical form (CCF) defined in Chapter 1 and the answer is not unique. Many authors call this the Kalman controllability decomposition. (Hint: To simplify the matrix inversion, you can use elementary column operations to come up with a simple basis for the column span of $\mathcal{C}$ rather than using a set of linearly independent columns.)

Solution: The controllability matrix is $\mathcal{C} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ -2 & -3 & -5 \end{bmatrix}$. Using elementary column operations (start by subtracting the first column from the second and third columns) we find $\mathcal{C}$ has the same column span as $\mathcal{C}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$. The controllable subspace is spanned by the columns of $\mathcal{C}$ and thus by the columns of $\mathcal{C}'$, and it has dimension two. To select a choice of P^{-1} we use the first two columns of $\mathcal{C}'$ and add a third column not in the span of the first two. (Other choices are possible, such as using the first two columns of $\mathcal{C}$.) Specifically, we use $P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ which has inverse $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. We then find

$$\bar{A} = PAP^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \bar{B} = PB = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

which is in Kalman controllability canonical form. Specifically the third row first two entries of $\bar{A}$ and third entry in $\bar{B}$ are zeros, and the 2×2 leading minor of $\bar{A}$ and first two entries in $\bar{B}$ correspond to a controllable second order system, i.e. $A_c = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. and $B_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; it happens to be in modal form.

NOTE: If instead we had taken the first two columns of P^{-1} to be the first two columns of $\mathcal{C}$ and the third to be $[0\ 0\ 1]^T$ then we would have found the following version of KCCF:

$$\bar{A} = PAP^{-1} = \begin{bmatrix} 0 & -2 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \bar{B} = PB = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Note that both solutions have the same eigenvalue for the uncontrollable part, namely 1. And the eigenvalues for the controllable parts are 1 and 2 in both cases. Other choices of P are possible giving yet other versions of the KCCF but they will all be similar in the above senses.

5. **[Minimal realization of a MIMO transfer function]**
Using the matrix partial fraction expansion described at the end of Chapter 6, find a minimal state-space representation for the 3×2 transfer function

$$P(s) = \frac{1}{(s+2)(s+3)(s+4)} \begin{bmatrix} 2s^2+13s+20 & s^2+7s+12 \\ s^2+5s+6 & 2s^2+12s+18 \\ 2s^2+11s+14 & s^2+5s+6 \end{bmatrix}$$

Solution: By partial fraction expansion of each of the six entries in $P(s)$ and suitably arranging, we find

$$\begin{aligned} P(s) &= \frac{1}{s+2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{s+3} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + \frac{1}{s+4} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{s+2} R_1 + \frac{1}{s+3} R_2 + \frac{1}{s+4} R_3 \end{aligned}$$

The sum of the ranks of R_1, R_2, R_3 is 2 + 1 + 1 =4. So any realization of $P(s)$ with a four dimensional state space is minimal. For example, taking

$$\begin{aligned} R_1 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = C_1 B_1 \\ R_2 &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = C_2 B_2 \\ R_3 &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = C_3 B_3, \end{aligned}$$

we can write

$$
\begin{aligned}
P(s) &= [C_1|C_2|C_3] \begin{bmatrix} \frac{1}{s+2} & 0 & 0 \\ 0 & \frac{1}{s+3} & 0 \\ 0 & 0 & \frac{1}{s+4} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+2} & 0 & 0 & 0 \\ 0 & \frac{1}{s+2} & 0 & 0 \\ 0 & 0 & \frac{1}{s+3} & 0 \\ 0 & 0 & 0 & \frac{1}{s+4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \\
&= C(sI - A)^{-1}B.
\end{aligned}
$$

where

$$
C = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}
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