University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 6
Observability, Duality, and Minimality

Due: Wednesday, October 16, 11:59pm
Reading: Course notes, Chapter 6

1. **[Adjoint linear system]**
Let ϕ denote the state transition matrix for the LTV system $\dot{x} = A(t)x$ in $\mathbb{R}^n$ and let ϕ_a denote the state transition matrix for the associated *adjoint system* defined by $\dot{z} = -A^* z$.)

   (a) Find an expression for $\frac{d}{dt}\phi(t_0, t)$ by using the fact $\frac{d}{dt}\{\phi(t_0, t)\phi(t, t_0)\} = 0_{n\times n}$ and removing the common factor $\phi(t, t_0)$.

   (b) Show that $\phi_a(t, t_0) = \phi^*(t_0, t)$. (Hint: For t_0 fixed, $\phi(t, t_0)$ is determined for all t by $\frac{d}{dt}\phi(t, t_0) = A(t)\phi(t, t_0)$ along with $\phi(t_0, t_0) = I$. Similarly, $\phi_a(t, t_0)$ is determined by $\frac{d}{dt}\phi_a(t, t_0) = -A^*(t)\phi_a(t, t_0)$ along with $\phi_a(t_0, t_0) = I$.)

   (c) Show that $z^*(t)x(t) = z^*(0)x(0)$ for all t in two different ways: (i) by differentiation and (ii) using the state transition matrices.

2. **[Classification of first order LTI systems]**
Consider the first-order SISO LTI system model

$$\dot{x} = ax + bu$$
$$y = cx + du.$$

   (a) Under what conditions on a, b, c, d is the system controllable?

   (b) Under what conditions on a, b, c, d is the system observable?

   (c) Under what conditions on a, b, c, d is the system internally asymptotically stable (i.e. if there is zero control the state converges to zero from any initial condition)?

   (d) Under what condition on a, b, c, d is the system a minimal realization?

3. **[Some realizations that are not minimal]**
Consider the transfer function

$$P(s) = \frac{s+2}{(s+2)(s+5)} = \frac{1}{s+5}.$$

   (a) Obtain the second order state space realization in controllable canonical form (CCF). Is it controllable? Is it observable?

   (b) Obtain the second order state space realization in observable canonical form (OCF). Is it controllable? Is it observable?

   (c) Obtain the second order state space realization in modal form. Is it controllable? Is it observable?

4. **[Kalman controllability decomposition]**
Consider the LTI state space system $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & -3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

Find a nonsingular matrix P so that the corresponding change of coordinates brings the system into the Kalman controllability canonical form as in Chapter 5. This is not the same as the controllable canonical form (CCF) defined in Chapter 1 and the answer is not unique. Many authors call this the Kalman controllability decomposition. (Hint: To simplify the matrix inversion, you can use elementary column operations to come up with a simple basis for the column span of $\mathcal{C}$ rather than using a set of linearly independent columns.)

5. **[Minimal realization of a MIMO transfer function]**
Using the matrix partial fraction expansion described at the end of Chapter 6, find a minimal state-space representation for the 3×2 transfer function

$$P(s) = \frac{1}{(s+2)(s+3)(s+4)} \begin{bmatrix} 2s^2 + 13s + 20 & s^2 + 7s + 12 \\ s^2 + 5s + 6 & 2s^2 + 12s + 18 \\ 2s^2 + 11s + 14 & s^2 + 5s + 6 \end{bmatrix}$$