ECE 515/ME 540: Problem Set 5: Problems and Solutions
Controllability

Due: Wednesday, October 9, 11:59pm
Reading: Course notes, Chapter 5

1. **[Controllability properties for some LTI systems]**
For each of the A, B pairs below determine whether the LTI system $\dot{x} = Ax + Bu$ is controllable. For the ones that are not controllable, find the controllable subspace Σ_c and also find an eigenvalue λ of A such that the Hautus-Rosenbrock test for controllability fails.

(a)

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution:

$$\mathcal{C} = \begin{bmatrix} 1 & 6 & 35 \\ 1 & 5 & 25 \\ 1 & 2 & 4 \end{bmatrix}$$

Controllable. (To see $\mathcal{C}$ is full rank we could subtract first row from each of the other two rows and then do one more elementary row operation.) BTW, A consists of two Jordan blocks with different eigenvalues.

(b)

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution:

$$\mathcal{C} = \begin{bmatrix} 1 & 6 & 36 \\ 1 & 6 & 35 \\ 1 & 5 & 25 \end{bmatrix}$$

Controllable. (To see $\mathcal{C}$ is full rank we could subtract first row from each of the other two rows and see it is full rank.) BTW, A consists of a single Jordan block.

(c)

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution:

$$\mathcal{C} = \begin{bmatrix} 1 & 6 & 35 \\ 1 & 5 & 25 \\ 1 & 5 & 25 \end{bmatrix}$$

Not controllable. $\mathcal{C}$ has rank 2 – the row span has dimension two. Columns 1,2 span. $\Sigma_s = \{x \in \mathbb{R}^3 : x_2 = x_3\}$. The Hautus-Rosenbrock test fails for $\lambda = 5$ because

$$[(5I - A) \;\; B] = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which does not have full rank. BTW, A consists of two Jordan blocks with the same eigenvalue. There does not exist a choice of column vector B that could make the system controllable by the Hautus-Rosenbrock test.

(d)

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 \\ 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Solution:

$$\mathcal{C} = \begin{bmatrix} 4 & 0 & 0 & 2 & 0 & 4 & 0 & 8 \\ 2 & 0 & 0 & 2 & 0 & 4 & 0 & 8 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 8 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 8 \end{bmatrix}$$

Not controllable; the row span has dimension three. Columns 1,2,4 span. $\Sigma_s = \{x \in \mathbb{R}^4 : x_3 = x_4\}$. Since A is upper triangular the eigenvalues are the diagonal entries. The Hautus-Rosenbrock test fails for $\lambda = 1$ because

$$[(I - A) \;\; B] = \begin{bmatrix} 1 & 0 & -1 & -1 & 4 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

which is singular (third and fourth rows are the same).

2. **[Controllability for a linear system that is piecewise time invariant]** Consider the time-varying system $\dot{x} = A(t)x + Bu$ over the time interval $0 \leq t \leq 2$, where

$$(A(t), B(t)) = \begin{cases} (A_1, B_1) & \text{if } 0 \leq t < 1 \\ (A_2, B_2) & \text{if } 1 \leq t \leq 2 \end{cases} \tag{1}$$

and (A_1, B_1) and (A_2, B_2) are the matrices for two LTI systems with the same dimensions. We consider controllability of the time-varying system for the two fixed times $t_0 = 0$ and $t_f = 2$. That is, taking any state x_o at time $t = 0$ to any state x_f at time $t = 2$.

(a) Find $\phi(0,\tau)$ for $0 \le \tau \le 2$. (Hint: This is the state transition matrix for going backwards in time from τ to 0.)

Solution:

$$\phi(0,\tau) = \begin{cases} e^{-A_1\tau} & \text{if } 0 \le \tau < 1 \\ \phi(0,1)\phi(1,\tau) = e^{-A_1}e^{-A_2(\tau-1)} & \text{if } 1 \le \tau \le 2 \end{cases}$$

Note that since we do not assume that $A_1A_2 = A_2A_1$ it would be incorrect to write $\phi(0,\tau) = e^{-(A_1+A_2(\tau-1))}$ for $1 \le \tau \le 2$.

(b) Express the controllability Grammian matrix $W(0,2)$ in terms of the controllability Grammian matrices $W_1(0,1)$ and $W_2(0,1)$ for the two subsystems corresponding to (A_1, B_1) and (A_2, B_2).

Solution:

$$\begin{aligned} W(0,2) &:= \int_0^2 \phi(0,\tau)B(\tau)B^*(\tau)\phi^*(0,,\tau)d\tau \\ &= \int_0^1 e^{A_1\tau}B_1B_1^*e^{A_1^*\tau}d\tau + \int_1^2 e^{-A_1}e^{-A_2(\tau-1)}B_2B_2^*e^{-A_2^*(\tau-1)}e^{-A_1^*}d\tau \\ &= W_1(0,1) + e^{-A_1}\left(\int_0^1 e^{-A_2\tau}B_2B_2^*e^{-A_2^*\tau}d\tau\right)e^{-A_1^*} \\ &= W_1(0,1) + e^{-A_1}W_2(0,1)e^{-A_1^*} \end{aligned}$$

(c) Is it true or false that $(A(t), B(t))_{0\le t\le 2}$ is controllable if either (A_1, B_1) or (A_2, B_2) is controllable? Justify your answer. (Hint: Controllability Grammians are always Hermitian positive semi-definite matrices.)

Solution: True. For any nonzero vector α,

$$\alpha^*W(0,2)\alpha = \alpha^*W_1(0,1)\alpha + \alpha^*e^{-A_1}W_2(0,1)e^{-A_1^*}\alpha \tag{2}$$

and both terms on the righthand side of (2) are nonnegative. If (A_1, B_1) is controllable then the first term is strictly positive and if (A_2, B_2) is controllable then the second term is strictly positive, using the fact e^{-A_1} is nonsingular. So if at least one of the two subsystems is controllable then so is the composite system.

(d) Is it true or false that $(A(t), B(t))_{0\le t\le 2}$ is controllable *only* if either (A_1, B_1) or (A_2, B_2) is controllable? Justify your answer.

Solution: False. For example, let

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Intuitively, x_1 can be controlled during $[0,1]$ and x_2 can be controlled over $[1,2]$. For this choice

$$W_1(0,1) = \begin{bmatrix} \frac{e^2-1}{2} & 0 \\ 0 & 0 \end{bmatrix} \quad W_2(0,1) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{e^2-1}{2} \end{bmatrix} \quad W(0,2) = \begin{bmatrix} \frac{e^2-1}{2} & 0 \\ 0 & \frac{e^2-1}{2} \end{bmatrix}.$$

3. **[Generalized matrix inverse]**

Suppose $A \in \mathcal{M}_{n,m}(\mathbb{C})$ (i.e. A is an $n \times m$ matrix with complex entries) and b is an n vector. Suppose $n < m$ and consider solutions u to the linear system of equations $Au = b$.

(a) Show that there is a solution for all $b \in \mathbb{C}^n$ if and only if A has full rank (i.e. rank n).

Solution: There are solutions for all b if and only if the column span of A is equal to $\mathbb{C}^n$, or equivalently, if and only if the dimension of the column span of A is n, or equivalently, if and only if A has rank n.

(b) Show that A has full rank if and only if the Grammian matrix AA^* has full rank.

Solution: If A does not have full rank then its rows are not linearly independent so there exists a nonzero vector $\alpha \in \mathbb{C}^n$ such that $\alpha^* A = \vartheta_{m\times 1}$. That implies that $\alpha^*(AA^*) = \vartheta_{1\times n}$ so AA^* is not full rank. Conversely, if AA^* is not full rank then there exists a nonzero vector α such that $\alpha^*(AA^*) = \vartheta_{n\times 1}$. But then $\|\alpha^* A\|^2 = \alpha^* AA^* \alpha = 0$ so $\alpha^* A = \vartheta_{1\times m}$ so A does not have full rank.

(c) Suppose A has full rank. Then a solution of the linear equation $Au = b$ is given by $u = A^*(AA^*)^{-1}b$. Let $\tilde{u}$ denote another solution, so $A\tilde{u} = b$. Show that $\|\tilde{u}\|^2 = \|\tilde{u} - u\|^2 + \|u\|^2$. Therefore, $u = A^*(AA^*)^{-1}b$ is the solution to $Au = b$ with the minimum norm. (Hint: $A(\tilde{u} - u) = b - b = \vartheta_{n\times 1}$.) The matrix $A^*(AA^*)^{-1}$ is called the *generalized inverse* of A.)

Solution: We have $\tilde{u} = (\tilde{u} - u) + u$ and $\langle u, \tilde{u} - u\rangle = u^*(\tilde{u} - u) = b^*(AA^*)^{-1}A(\tilde{u} - u) = b^*(AA^*)^{-1}\vartheta = 0$. In other words, $\tilde{u}$ is the sum of the orthogonal vectors $\tilde{u} - u$ and u. Therefore, $\|\tilde{u}\|^2 = \|\tilde{u} - u\|^2 + \|u\|^2$ as claimed.

4. **[Controllable canonical form]**
Suppose (A, B) is a controllable pair such that B is an $n \times 1$ matrix (so single input system) and let $\mathcal{C}$ denote its controllability matrix. Write the characteristic polynomial of A as $\triangle(s) := \det(sI - A) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$. The *controllable canonical form* (CCF) for the characteristic polynomial $\triangle(s)$ is the pair $(\bar{A}, \bar{B})$ given by

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

(The α's here are indexed as in Chapter 7 of the class notes – not the same as in Chapter 1.)

(a) Show that the controllability matrix $\bar{\mathcal{C}}$ for $(\bar{A}, \bar{B})$ is full rank – this shows that the CCF is indeed controllable. Also, explain why $\bar{A}$ also has characteristic polynomial $\triangle(s)$.

Solution: By induction on k, reading down the entries of the k^{th} column of $\bar{\mathcal{C}}$ we find $n - k$ 0's followed by a 1 followed by other more complicated terms. So we have that $\bar{\mathcal{C}}$ has the form:

$$\bar{\mathcal{C}} = \left[\bar{B}\ \bar{A}\bar{B}\ \bar{A}^2\bar{B}\ \cdots\ \bar{A}^{n-1}\bar{B}\right] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & * & * \\ 1 & * & \cdots & * & * \end{bmatrix}$$

where the * terms depend on the α's. The matrix is nonsingular (because the columns are clearly linearly independent or because $\det(C) = (-1)^n \neq 0$).

The characteristic polynomial of $\bar{A}$ is given by

$$\det(sI - \bar{A}) = \det \begin{bmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & s & -1 \\ \alpha_n & \alpha_{n-1} & \cdots & \cdots & s+\alpha_1 \end{bmatrix}.$$

We will prove that $\det(sI - \bar{A}) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$ by induction on n. The result is true for the base case $n = 1$: in that case both are equal to $s + \alpha_1$. Suppose the equality is true for all order $n-1$ CCF systems for all choices of the coefficients. Given an order n system, we evaluate the determinant by *expansion by minors* down the first column. Let $M = sI - \bar{A}$ and let $M^{i,j}$ denote the $(i,j)^{th}$ minor of M, obtained by deleting the i^{th} row and j^{th} column. Then the determinant is $\sum_{i=1}^{n} (-1)^{i+1} m_{i,1} \det(M^{i,1})$. There are two nonzero terms in the sum. Since $M^{1,1}$ is also a matrix in the CCF form and $M^{n,1} = -I_{(n-1)\times(n-1)}$, we get by the induction hypothesis that

$$\begin{aligned} \det(sI - \bar{A}) &= s\det(M^{1,1}) + (-1)^{n+1}\alpha_n \det(M^{n,1}) \\ &= s\left(s^{n-1} + \alpha_1 s^{n-2} + \cdots + \alpha_{n-2}s + \alpha_{n-1}\right) + (-1)^{n+1}\alpha_n(-1)^{n-1} \\ &= s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n \end{aligned}$$

as claimed.

ALTERNATIVE DERIVATION: To evaluate the determinant we use the formula involving a sum over permutations (see Problem set 2, Problem 3(d)). The identity permutation gives the terms $s^n + \alpha_1 s^{n-1}$. A little thought shows that for each k with $1 \le k \le n-1$ there is only one permutation that gives a nonzero term in the sum that includes α_{n-k}. That term is given by

$$\det \underbrace{\begin{bmatrix} s & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & s & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & s & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & \cdots & 0 & \alpha_{n-k} & \cdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{k}$$

Starting from the matrix shown we can repeatedly swap the column containing α_{n-k} with the column to its right while changing the -1 in the column moving left to +1. Such swaps don't change the determinant and result in the determinant:

$$\det \begin{bmatrix} s & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & s & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ \underbrace{0 \quad 0 \quad \cdots \quad 0}_{k} & & & & 0 & \cdots & 0 & 0 & 0 & \alpha_{n-k} \end{bmatrix} = \alpha_{n-k} s^k$$

Hence $\det(sI - \bar{A}) = \triangle(s)$ as claimed.

(b) Show that $A\mathcal{C} = \mathcal{C}\bar{A}^T$ and $\mathcal{C}^{-1}B = e^1$, where $e^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Solution: We find that

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} B & AB & \cdots & A^{n-2}B & A^{n-1}B \end{bmatrix} \\ A\mathcal{C} &= \begin{bmatrix} AB & A^2B & \cdots & A^{n-1}B & A^nB \end{bmatrix} \\ \bar{A}^T &= \begin{bmatrix} 0 & 0 & 0 & \cdots & -\alpha_n \\ 1 & 0 & 0 & \cdots & -\alpha_{n-1} \\ 0 & 1 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & -\alpha_2 \\ 0 & 0 & \cdots & 1 & -\alpha_1 \end{bmatrix} \end{aligned}$$

and by the Cayley-Hamilton theorem, $A^n = -\alpha_1 A^{n-1} + \cdots - \alpha_{n-1}A - \alpha_n I$, from which the equation $A\mathcal{C} = \mathcal{C}\bar{A}^T$ follows.

It is easily checked that $\mathcal{C}e^1 = B$ from which it follows that $\mathcal{C}^{-1}B = e^1$.

(c) Explain how to use part (b) to quickly show: $\bar{A}\bar{\mathcal{C}} = \bar{\mathcal{C}}\bar{A}^T$ and $\bar{\mathcal{C}}^{-1}\bar{B} = e^1$.

Solution: Replace (A, B) by $(\bar{A}, \bar{B})$ in the equations of part (b). Since A and $\bar{A}$ have the same characteristic polynomial $\triangle(s)$, the equations hold as before.

(d) If $(\bar{A}, \bar{B})$ were equivalent to (A, B) under the change of coordinates $\bar{x} = Px$, then it must be that $\bar{\mathcal{C}} = P\mathcal{C}$ or $P = \bar{\mathcal{C}}\mathcal{C}^{-1}$. Show that for this P that $\bar{A} = PAP^{-1}$ and $\bar{B} = PB$. This proves that the original controllable pair (A, B) is equivalent to $(\bar{A}, \bar{B})$ under the change of coordinates.

Solution: Since $P^{-1} = \mathcal{C}\bar{\mathcal{C}}^{-1}$, the inequalties to be shown are:

$$\bar{A} = \bar{\mathcal{C}}\mathcal{C}^{-1}A\mathcal{C}\bar{\mathcal{C}}^{-1} \quad \text{and} \quad \bar{B} = \bar{\mathcal{C}}\underbrace{\mathcal{C}^{-1}B}_{e^1}.$$

or equivalently:

$$\bar{\mathcal{C}}^{-1}\underbrace{\bar{A}\bar{\mathcal{C}}}_{\bar{\mathcal{C}}\bar{A}^T} = \mathcal{C}^{-1}\underbrace{A\mathcal{C}}_{\mathcal{C}\bar{A}^T} \quad \text{and} \quad \bar{B} = \bar{\mathcal{C}}e^1$$

or equivalently (keeping in mind what $\bar{B}$ and $\bar{\mathcal{C}}$ are):

$$\bar{A}^T = \bar{A}^T \quad \text{and} \quad \bar{B} = \bar{B},$$

which is true.