University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 5
Controllability

Due: Wednesday, October 9, 11:59pm
Reading: Course notes, Chapter 5

1. **[Controllability properties for some LTI systems]**
For each of the A, B pairs below determine whether the LTI system $\dot{x} = Ax + Bu$ is controllable. For the ones that are not controllable, find the controllable subspace Σ_c and also find an eigenvalue λ of A such that the Hautus-Rosenbrock test for controllability fails.

   (a)
$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   (b)
$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   (c)
$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   (d)
$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 0 \\ 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

2. **[Controllability for a linear system that is piecewise time invariant]**
Consider the time-varying system $\dot{x} = A(t)x + Bu$ over the time interval $0 \leq t \leq 2$, where

$$(A(t), B(t)) = \begin{cases} (A_1, B_1) & \text{if } 0 \leq t < 1 \\ (A_2, B_2) & \text{if } 1 \leq t \leq 2 \end{cases} \tag{1}$$

and (A_1, B_1) and (A_2, B_2) are the matrices for two LTI systems with the same dimensions. We consider controllability of the time-varying system for the two fixed times $t_0 = 0$ and $t_f = 2$. That is, taking any state x_o at time $t = 0$ to any state x_f at time $t = 2$.

   (a) Find $\phi(0, \tau)$ for $0 \leq \tau \leq 2$. (Hint: This is the state transition matrix for going backwards in time from τ to 0.)

(b) Express the controllability Grammian matrix $W(0,2)$ in terms of the controllability Grammian matrices $W_1(0,1)$ and $W_2(0,1)$ for the two subsystems corresponding to (A_1, B_1) and (A_2, B_2).

(c) Is it true or false that $(A(t), B(t))_{0 \le t \le 2}$ is controllable if either (A_1, B_1) or (A_2, B_2) is controllable? Justify your answer. (Hint: Controllability Grammians are always Hermitian positive semi-definite matrices.)

(d) Is it true or false that $(A(t), B(t))_{0 \le t \le 2}$ is controllable *only* if either (A_1, B_1) or (A_2, B_2) is controllable? Justify your answer.

3. **[Generalized matrix inverse]**
Suppose $A \in \mathcal{M}_{n,m}(\mathbb{C})$ (i.e. A is an $n \times m$ matrix with complex entries) and b is an n vector. Suppose $n < m$ and consider solutions u to the linear system of equations $Au = b$.

   (a) Show that there is a solution for all $b \in \mathbb{C}^n$ if and only if A has full rank (i.e. rank n).

   (b) Show that A has full rank if and only if the Grammian matrix AA^* has full rank.

   (c) Suppose A has full rank. Then a solution of the linear equation $Au = b$ is given by $u = A^*(AA^*)^{-1}b$. Let $\tilde{u}$ denote another solution, so $A\tilde{u} = b$. Show that $\|\tilde{u}\|^2 = \|\tilde{u} - u\|^2 + \|u\|^2$. Therefore, $u = A^*(AA^*)^{-1}b$ is the solution to $Au = b$ with the minimum norm. (Hint: $A(\tilde{u} - u) = b - b = \vartheta_{n \times 1}$.) The matrix $A^*(AA^*)^{-1}$ is called the *generalized inverse* of A.)

4. **[Controllable canonical form]**
Suppose (A, B) is a controllable pair such that B is an $n \times 1$ matrix (so single input system) and let $\mathcal{C}$ denote its controllability matrix. Write the characteristic polynomial of A as $\triangle(s) := \det(sI - A) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$. The *controllable canonical form* (CCF) for the characteristic polynomial $\triangle(s)$ is the pair $(\bar{A}, \bar{B})$ given by

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

(The α's here are indexed as in Chapter 7 of the class notes – not the same as in Chapter 1.)

   (a) Show that the controllability matrix $\bar{\mathcal{C}}$ for $(\bar{A}, \bar{B})$ is full rank – this shows that the CCF is indeed controllable. Also, explain why $\bar{A}$ also has characteristic polynomial $\triangle(s)$.

   (b) Show that $A\mathcal{C} = \mathcal{C}\bar{A}^T$ and $\mathcal{C}^{-1}B = e^1$, where $e^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

   (c) Explain how to use part (b) to quickly show: $\bar{A}\bar{\mathcal{C}} = \bar{\mathcal{C}}\bar{A}^T$ and $\bar{\mathcal{C}}^{-1}\bar{B} = e^1$.

   (d) If $(\bar{A}, \bar{B})$ were equivalent to (A, B) under the change of coordinates $\bar{x} = Px$, then it must be that $\bar{\mathcal{C}} = P\mathcal{C}$ or $P = \bar{\mathcal{C}}\mathcal{C}^{-1}$. Show that for this P that $\bar{A} = PAP^{-1}$ and $\bar{B} = PB$. This proves that the original controllable pair (A, B) is equivalent to $(\bar{A}, \bar{B})$ under the change of coordinates.