University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 4: Problems and Solutions Stability

Due: Wednesday, September 25, 11:59pm
Reading: Course notes, Chapter 4

1. **[Stability of systems]**
Determine the equilibrium points of the following three dynamical systems and for each equilibrium point, determine in which of the following three senses the equilibrium point is stable: LS - Lyapunov stability, AS - asymptotic stability, GAS – global asymptotic stability. Refer directly to the definitions of stability without using tools such as eigenvector analysis or Lyapunov functions. The systems evolve in real Euclidean spaces $\mathbb{R}^2, \mathbb{R}^2$, and $\mathbb{R}$, respectively.

$$(a)\ \begin{array}{l} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -x_2(1 - x_1^2) \end{array} \qquad (b)\ \begin{array}{l} \dot{x}_1 = 0 \\ \dot{x}_2 = -x_1^2 \end{array} \qquad (c)\ \dot{x} = x(x-1)(x-2)$$

Solution: (a) Setting $\dot{x}_1 = \dot{x}_2 = 0$ to identify the equilibrium points we find that $x_e = \vartheta$ (the origin) is the only equilibrium point. The evolution of x_1 does not depend on x_2 and its vector field takes it monotonically to zero. After the first time that $|x_1| < r$ for some constant r with $0 < r < 1$ then x_2 converges monotonically to 0 as well. So x_e is GAS (globally asymptotically stable, and hence also LS and AS.

Solution: (b) Any x such that $x_1 = 0$ is an equilibrium point. Inside a small disk centered around any equilibrium point, there is a possible initial state x with $x_1 = c$ for some small, nonzero constant c. Then $x_1(t) = c$ for all t and $\dot{x}_2 = -c^2$ for all t. Hence $x_2(t) \to -\infty$ as $t \to \infty$. Therefore, none of the equilibria are stable in any of the three senses.

Solution: (c) The points $x = 0, 1, 2$ are the equilibrium points. Note that $\dot{x} > 0$ for $0 < x < 1$ or $x > 2$ and $\dot{x} < 0$ for $x < 0$ and $1 < x < 1$. This can be represented by the following one dimensional vector field (aka phase diagram):

$$\cdots \leftarrow\leftarrow\leftarrow\leftarrow \mathbf{0} \rightarrow\rightarrow\rightarrow\rightarrow \mathbf{1} \leftarrow\leftarrow\leftarrow\leftarrow \mathbf{2} \rightarrow\rightarrow\rightarrow\rightarrow \cdots .$$

From this we see that equilibrium points 0 and 2 are unstable in all three senses, and equilibrium point 1 is locally asymptotically stable (and hence also Lyapunov stable) but not globally asymptotically stable.

Note: The interval $(0, 2)$ is the maximal region of asymptotic stability for $x = 1$.

2. **[A region of asymptotic stability]**
Consider the dynamical system:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 + 2x_1^2. \end{aligned}$$

Use the Lyapunov function $V(x) = \frac{1}{2}(x_1^2 + x_2^2) = \frac{1}{2}\|x\|^2$ to determine a region of asymptotic stability for the equilibrium point $x_e = \vartheta$.

Solution: Calculate as follows:

$$\frac{d}{dt}V(x(t)) = \nabla V(x(t)) \cdot f(x(t)) = x \cdot \begin{bmatrix} -x_1 + x_2^2 \\ -x_2 + 2x_1^2 \end{bmatrix} = -x_1^2(1 - 2x_2) - x_2^2(1 - x_1).$$

Consider the sublevel set of V for level $\ell = 1/8$:

$$\Omega_{1/8} := \left\{ x : V(x) < \frac{1}{8} \right\} = \left\{ x : \|x\| < \frac{1}{2} \right\}$$

For $x \in \Omega_{1/8}$, $x_2 \le \|x\| < \frac{1}{2}$ so that $1 - 2x_2 > 0$ and also $1 - x_1 > 0$. So $\frac{d}{dt}V(x(t)) < 0$ if $x(t) \in \Omega_{1/8}$. It follows from Theorem 4.4 in the course notes that $\Omega_{1/8}$ is a region of asymptotic stability for $x_e = \vartheta$. (The value $\ell = 1/8$ is not the largest value of ℓ that works – it would require more work to find that. Any smaller positive value of ℓ would also give a region of asymptotic stability.)

3. **[Stability of a pendulum]**
The dynamical system of a pendulum is given by $\ddot{\theta} = -b\dot{\theta} - \sin(\theta)$ where θ is the angle between the position of the pendulum and straight down and b represents a damping force such as mild air resistance. We investigate the stability of the equilibrium point $\theta = \dot{\theta} = 0$.

(a) The system energy (kinetic plus potential energy) is given by $V(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + 1 - \cos(\theta)$. *For this part, assume that* $b = 0$. Use the Lyapunov direct method (aka second method) with V as the Lyapunov function to see what the method implies about the senses (LS, AS, GAS), if any, in which the system is stable. For consistency, use the coordinates x with $x_1 = \theta$ and $x_2 = \dot{\theta}$.

Solution: The system dynamics is $\dot{x} = f(x)$ where $f(x) = \begin{bmatrix} x_2 \\ -bx_2 - \sin(x_1) \end{bmatrix}$ and the Lyapunov function is $V(x) = \frac{1}{2}x_2^2 + 1 - \cos(x_1)$.

$$\frac{d}{dt}V(x(t)) = \nabla V(x(t)) \cdot f(x(t)) = -bx_2^2(t). \tag{1}$$

For this problem part we assume $b = 0$ so that $\frac{d}{dt}V(x(t)) = 0$. It follows that $\frac{d}{dt}V(x(t)) \le 0$ in a neighborhood of $x_e = \vartheta$. Therefore, it follows that x_e is Lyapunov stable.
Note: Since $\frac{d}{dt}V(x(t)) = 0$ for all t it follows that $V(x(t)) = V(x_0)$ for all $t \ge 0$ so if $x(0) \ne x_e$ then $x(t) \not\to x_e$. Therefore x_e is not asymptotically stable.

(b) Repeat part (a) but now assume $b > 0$.
Solution: By the calculation in (1), $\frac{d}{dt}V(x(t)) = -bx_2^2 \le 0$ from which we can again conclude that x_e is Lyapunov stable. The function bx_2^2 is positive semidefinite but not positive definite so we cannot conclude that x_e is asymptotically stable using this choice of V.
Note: A phase plot of this system shows that x_e is asymptotically stable. One way to prove it by the direct Lyapunov method is to use a different Lyapunov function with the term x_2^2 replaced by a suitable quadratic function of x depending on both coordinates. Another is to invoke Lasalle's invariance principle (beyond the scope of the course) to first conclude that the distance of $x(t)$ to the set where $bx^2(t) = 0$ converges to zero.

(c) Assuming $b > 0$ as in part (b), examine the linear dynamical system obtained by linearizing the dynamics around the equilibrium point $x_e = \vartheta$. What can you conclude about the stability of x_e for the original system (with $b > 0$) from properties of the linear system?
Solution: The linearized system is $\dot{\delta x} = A\delta x$ where

$$A = \left.\frac{\partial f}{\partial x}\right|_{x_e} = \left.\begin{bmatrix} 0 & 1 \\ -\cos(x) & -b \end{bmatrix}\right|_{x_e} = \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix}$$

The eigenvalues of A are $\frac{-b\pm\sqrt{b^2-4}}{2}$, which both have negative real parts for any $b > 0$. Therefore $x_e = \vartheta$ is an asymptotically stable equilibrium (by Theorem 4.7 in the course notes). From this analysis we cannot conclude global stability.

Note that if $0 < 2 < b$ then the eigenvalues have a nonzero imaginary part, meaning the pendulum will repeatedly swing past the equilibrium point while settling down to coverge, while if $b \geq 2$ the damping force is so strong that the pendulum monotonically converges to its equilibrium. Further analysis of this problem shows that the equilibrium is globally asymptotically stable with the understanding that angles differing by multiples of 2π are the same. That is becasue if the initial velocity is large enough, the pendulum could spin around multiple times before slowing down and settling into an equilibrium.

4. **[Stable invariant subspaces]**
This problem is aimed at understanding the stable invariant subspaces for LTI systems $\dot{x} = Ax$. A change of coordinates can reduce an $n \times n$ matrix A to Jordan canonical form, so we exam the case of a Jordan matrix J with a single Jordan block.

(a) Let $\lambda \in \mathbb{C}$ and find e^{Jt} for the 3×3 Jordan block matrix $J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$.

Solution: By induction on k we find that $J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$, where $\binom{k}{2}$ is the binomial coefficient equal to zero for $k = 0, 1$ and equal to $\frac{k(k-1)}{2}$ for $k \geq 2$. Then,

$$\begin{aligned} e^{Jt} &= \sum_{k=0}^{\infty} \frac{J^k t^k}{k!} \\ &= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!} & \sum_{k=1}^{\infty} \frac{k\lambda^{k-1}t^k}{k!} & \frac{1}{2}\sum_{k=2}^{\infty} \frac{k(k-1)\lambda^{k-2}t^k}{k!} \\ 0 & \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!} & \sum_{k=1}^{\infty} \frac{k\lambda^{k-1}t^k}{k!} \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!} \end{bmatrix} \\ &= \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}. \end{aligned}$$

(b) What is the necessary and sufficient condition on $\lambda \in \mathbb{C}$ such that $e^{Jt} \to 0_{3\times 3}$ as $t \to \infty$? (Note that for each $k \times k$ Jordan block in the canonical representation for a matrix A there corresponds an eigenvector v^1 and $k-1$ generalized eigenvectors $v^2, \ldots v^k$ such that $(I\lambda - A)v^j = v^{j-1}$ for $2 \leq j \leq k$. The span of $v^1, \ldots, v^k$ is an invariant subspace for the dynamics $\dot{x} = Ax$ and it is asymptotically stable if and only if $Re(\lambda) < 0$. The joint span corresponding to all Jordan blocks with $Re(\lambda) < 0$ is the stable invariant subspace for the dynamics.)

Solution: $Re(\lambda) < 0$. For example, if $Re(\lambda) = -a < 0$ then

$$|t^2 e^{\lambda t}| = t^2 e^{-at} = e^{-at+2\log t} \to 0. \tag{2}$$

(c) What is the necessary and sufficient condition on $\lambda \in \mathbb{C}$ such that e^{Jt} is bounded for all $t \geq 0$?

Solution: $Re(\lambda) < 0$. (If $Re(\lambda) = 0$ then the factors t and t^2 would make e^{Jt} unbounded as $t \to \infty$.)

5. **[Uniqueness of solution to Lyapunov equation for Hurwitz A]**
Suppose A is an $n \times n$ Hurwitz matrix, suppose Q is a postive-definite symmetric matrix, and suppose P_1, P_2 are both $n \times n$ matrix solutions to the Lyapunov equation $A^T P + PA = -Q$.

   (a) Show that $A^T(P_1 - P_2) + (P_1 - P_2)A = 0_{n\times n}$.
   Solution: It follows from: $(A^T P_1 + P_1 A) - (A^T P_2 + P_2 A) = -Q + Q = 0_{n\times n}$.

   (b) Show that $\frac{d}{dt}[e^{A^T t}(P_1 - P_2)e^{At}] = 0$. (Hint: $\dot{e^{At}} = Ae^{At} = e^{At}A$.)
   Solution: By the product rule for differentiation, part (a), and the hint:

$$\begin{aligned}\frac{d}{dt}[e^{A^T t}(P_1 - P_2)e^{At}] &= \dot{e^{A^T t}}(P_1 - P_2)e^{At} + e^{A^T t}(P_1 - P_2)\dot{e^{At}} \\ &= e^{A^T t}(A^T(P_1 - P_2) + (P_1 - P_2)A)e^{At}] \\ &= e^{A^T t}0_{n\times n}e^{At} = 0_{n\times n}.\end{aligned}$$

   (c) Examine the integral of the expression in part (b) over $t \in [0, \infty)$ to conclude that $P_1 = P_2$.
   Solution: By part (b) and the fundamental theorem of calculus,

$$\begin{aligned}0_{n\times n} &= \int_0^\infty \frac{d}{dt}[e^{A^T t}(P_1 - P_2)e^{At}]dt \\ &= e^{A^T t}(P_1 - P_2)e^{At}\Big|_0^\infty \\ &= 0_{n\times n} - (P_1 - P_2)\end{aligned}$$

   where we used the fact $\lim_{t\to\infty} e^{At} = \lim_{t\to\infty} e^{A^T t} = 0_{n\times n}$ by the assumption that A is a Hurwitz matrix. It follows that $P_1 = P_2$.

6. **[Lyapunov stability equation $A^T P + PA = -Q$]**
Consider the LTI system $\dot{x} = Ax$ for

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 0 \end{bmatrix}. \tag{3}$$

   (a) Directly determine if there is a unique solution to the Lyapunov stability equation $A^T P + PA = -I$ (so take $Q = I$), and if yes then see if the solution is symmetric and positive definite. Does your answer determine if the system is globally asymptotically stable? To be specific, assume that $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and start by identifying the set of linear equations for a, b, c, d.
   Solution: The equation $A^T P + PA = -I$ is equivalent to

$$\begin{bmatrix} -2a + 3b + 3c & -2a - b + 3d \\ -2a - c + 3d & -2b - 2c \end{bmatrix} = -I \tag{4}$$

   corresponding to the following augmented matrix in Gaussian elimination form:

$$\left[\begin{array}{cccc|c} -2 & 3 & 3 & 0 & -1 \\ -2 & -1 & 0 & 3 & 0 \\ -2 & 0 & -1 & 3 & 0 \\ 0 & -2 & -2 & 0 & -1 \end{array}\right] \tag{5}$$

yielding $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 15 \\ 3 \\ 3 \\ 11 \end{bmatrix}$ or $P = \begin{bmatrix} 5/4 & 1/4 \\ 1/4 & 11/12 \end{bmatrix}$. Since $p_{11} > 0$ and $\det(P) > 0$ we see that P is positive definite (by Sylvester's test – i.e. the leading principle minors have positive determinants). Therefore the system is asymptotically stable.

(b) Repeat part (a) for

$$A = \begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix} \tag{6}$$

Solution: Working as in part (a) we get

$$\begin{bmatrix} 6a - 5b - 5c & 5a + 6b - 5d \\ 5a + 6c - 5d & 5b + 5c + 6d \end{bmatrix} = -I. \tag{7}$$

Solving, we find there is a unique solution given by

$$P = \begin{bmatrix} -\frac{1}{6} & 0 \\ 0 & -\frac{1}{6} \end{bmatrix} \tag{8}$$

The matrix P is not positive definite, for example it fails Sylvester's criterion because the first principle minor, namely $-\frac{1}{6}$, has a negative determinant. Or we could note that both eigenvalues are negative. Therefore, we can conclude that the system is not asymptotically stable. (If it were we know there would exist a unique, symmetric, positive definite solution P for any positive definite choice of Q in the Lyapunov equation.)