University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 4
Stability

Due: Wednesday, September 25, 11:59pm
Reading: Course notes, Chapter 4

1. **[Stability of systems]**
Determine the equilibrium points of the following three dynamical systems and for each equilibrium point, determine in which of the following three senses the equilibrium point is stable: LS - Lyapunov stability, AS - asymptotic stability, GAS – global asymptotic stability. Refer directly to the definitions of stability without using tools such as eigenvector analysis or Lyapunov functions. The systems evolve in real Euclidean spaces $\mathbb{R}^2, \mathbb{R}^2$, and $\mathbb{R}$, respectively.

$$(a)\ \begin{array}{l}\dot{x}_1 = -x_1 \\ \dot{x_2} = -x_2(1-x_1^2)\end{array} \qquad (b)\ \begin{array}{l}\dot{x}_1 = 0 \\ \dot{x_2} = -x_1^2\end{array} \qquad (c)\ \dot{x} = x(x-1)(x-2)$$

2. **[A region of asymptotic stability]**
Consider the dynamical system:

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x_2} &= -x_2 + 2x_1^2.\end{aligned}$$

Use the Lyapunov function $V(x) = \frac{1}{2}(x_1^2 + x_2^2) = \frac{1}{2}\|x\|^2$ to determine a region of asymptotic stability for the equilibrium point $x_e = \vartheta$.

3. **[Stability of a pendulum]**
The dynamical system of a pendulum is given by $\ddot{\theta} = -b\dot{\theta} - \sin(\theta)$ where θ is the angle between the position of the pendulum and straight down and b represents a damping force such as mild air resistance. We investigate the stability of the equilibrium point $\theta = \dot{\theta} = 0$.

   (a) The system energy (kinetic plus potential energy) is given by $V(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + 1 - \cos(\theta)$. *For this part, assume that* $b = 0$. Use the Lyapunov direct method (aka second method) with V as the Lyapunov function to see what the method implies about the senses (LS, AS, GAS), if any, in which the system is stable. For consistency, use the coordinates x with $x_1 = \theta$ and $x_2 = \dot{\theta}$.

   (b) Repeat part (a) but now assume $b > 0$.

   (c) Assuming $b > 0$ as in part (b), examine the linear dynamical system obtained by linearizing the dynamics around the equilibrium point $x_e = \vartheta$. What can you conclude about the stability of x_e for the original system (with $b > 0$) from properties of the linear system?

4. **[Stable invariant subspaces]**
This problem is aimed at understanding the stable invariant subspaces for LTI systems $\dot{x} = Ax$. A change of coordinates can reduce an $n \times n$ matrix A to Jordan canonical form, so we exam the case of a Jordan matrix J with a single Jordan block.

(a) Let $\lambda \in \mathbb{C}$ and find e^{Jt} for the 3×3 Jordan block matrix $J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$.

(b) What is the necessary and sufficient condition on $\lambda \in \mathbb{C}$ such that $e^{Jt} \to 0_{3\times 3}$ as $t \to \infty$? (Note that for each $k \times k$ Jordan block in the canonical representation for a matrix A there corresponds an eigenvector v^1 and $k-1$ generalized eigenvectors $v^2, \ldots v^k$ such that $(I\lambda - A)v^j = v^{j-1}$ for $2 \le j \le k$. The span of $v^1, \ldots, v^k$ is an invariant subspace for the dynamics $\dot{x} = Ax$ and it is asymptotically stable if and only if $Re(\lambda) < 0$. The joint span corresponding to all Jordan blocks with $Re(\lambda) < 0$ is the stable invariant subspace for the dynamics.)

(c) What is the necessary and sufficient condition on $\lambda \in \mathbb{C}$ such that e^{Jt} is bounded for all $t \ge 0$?

5. **[Uniqueness of solution to Lyapunov equation for Hurwitz A]**
Suppose A is an $n \times n$ Hurwitz matrix, suppose Q is a postive-definite symmetric matrix, and suppose P_1, P_2 are both $n \times n$ matrix solutions to the Lyapunov equation $A^T P + PA = -Q$.

(a) Show that $A^T(P_1 - P_2) + (P_1 - P_2)A = 0_{n\times n}$.

(b) Show that $\frac{d}{dt}[e^{A^T t}(P_1 - P_2)e^{At}] = 0$. (Hint: $\dot{e^{At}} = Ae^{At} = e^{At}A$.)

(c) Examine the integral of the expression in part (b) over $t \in [0, \infty)$ to conclude that $P_1 = P_2$.

6. **[Lyapunov stability equation $A^T P + PA = -Q$]**
Consider the LTI system $\dot{x} = Ax$ for

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 0 \end{bmatrix}. \tag{1}$$

(a) Directly determine if there is a unique solution to the Lyapunov stability equation $A^T P + PA = -I$ (so take $Q = I$), and if yes then see if the solution is symmetric and positive definite. Does your answer determine if the system is globally asymptotically stable? To be specific, assume that $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and start by identifying the set of linear equations for a, b, c, d.

(b) Repeat part (a) for

$$A = \begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix} \tag{2}$$