University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 3: Problems and Solutions
Solutions of State Equations

Due: Wednesday, September 18, 11:59 pm
Reading: Course notes, Chapter 3

1. **[Computation of state transition matrix for an LTI system]**
The goal of this problem is for you to see how to compute e^{At} by hand for

$$A = \begin{bmatrix} -1 & 4 & 6 \\ 0 & -2 & 5 \\ 0 & 0 & -3 \end{bmatrix} \tag{1}$$

three different ways.

(a) Compute e^{At} by first finding the modal matrix and diagonalizing A. Work to completion.
Solution: We find $\triangle(s) = (s+1)(s+2)(s+3)$ and the eigenvalues are -1,-2,-3. A choice of the modal matrix, with eigenvectors as columns, is given by

$$M = \begin{bmatrix} 1 & -4 & 7 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

and we then find

$$M^{-1} = \begin{bmatrix} 1 & 4 & 13 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

and check that $AM = M\Lambda$ where

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}. \tag{4}$$

This gives

$$e^{At} = Me^{\Lambda t}M^{-1} = \begin{bmatrix} e^{-t} & 4e^{-t} - 4e^{-2t} & 13e^{-t} - 20e^{-2t} + 7e^{-3t} \\ 0 & e^{-2t} & 5e^{-2t} - 5e^{-3t} \\ 0 & 0 & e^{-3t} \end{bmatrix} \tag{5}$$

(b) Compute e^{At} by finding the Laplace transform of e^{At} and converting to the time domain. Carry out enough details to make sure you get the same answer as in part (a).
Solution: We find that

$$\mathcal{L}_t(e^{At})(s) = (Is - A)^{-1} = \begin{bmatrix} s+1 & -4 & -6 \\ 0 & s+2 & -5 \\ 0 & 0 & s+3 \end{bmatrix}^{-1} \tag{6}$$

$$= \begin{bmatrix} \frac{1}{s+1} & \frac{4}{(s+1)(s+2)} & \frac{6s+32}{(s+1)(s+2)(s+3)} \\ 0 & \frac{1}{s+2} & \frac{5}{(s+2)(s+3)} \\ 0 & 0 & \frac{1}{s+3} \end{bmatrix} \tag{7}$$

Using partial fraction expansions we find that the inverse Laplace transform gives the answer found in part (a).

(c) Show how e^{At} could be computed in this case by solving differential equations in the time domain to find the columns of e^{At}. This is possible due to the triangular structure of A. You do not need to carry out all details.

Solution: The idea is to solve $\dot{x} = Ax$ for an arbitrary initial condition at $t = 0$. Then the i^{th} column of e^{At} is the solution with initial condition e^i, where e^1, e^2, e^3 is the standard basis for $\mathbb{R}^3$. That is, we solve

$$\begin{aligned}\dot{x}_1 &= -x_1 + 4x_2 + 6x_3\\ \dot{x}_2 &= -2x_2 + 5x_3\\ \dot{x}_3 &= -3x_3\end{aligned}$$

The third equation involves only x_3 and is readily solved to give $x_3(t) = e^{-3t}x_3(0)$. Then the second equation can be solved to give

$$\begin{aligned}x_2(t) &= e^{-2t}x_2(0) + 5\int_0^t e^{-2(t-\tau)}x_3(\tau)d\tau\\ &= e^{-2t}x_2(0) + (5e^{-2t} - 5e^{-3t})x_3(0).\end{aligned}$$

Finally, the first equation can be solved to give

$$\begin{aligned}x_1(t) &= e^{-t}x_1(0) + 4\int_0^t e^{-(t-\tau)}(4x_2(\tau) + 6x_3(\tau))d\tau\\ &= e^{-t}x_1(0) + (4e^{-t} - 4e^{-2t})x_2(0) + (13e^{-t} - 20e^{-2t} + 7e^{-3t})x_3(0)\end{aligned}$$

We then see that the i^{th} column of e^{At} as found in parts (a) and (b) is given by the solution of $\dot{x} = Ax$ with initial condition $x(0) = e^i$ for $i = 1, 2, 3$.

2. **[Constant output linear system model]**
Consider a linear time invariant (LTI) system with zero input – so the matrices B and D are irrelevant.

(a) Let c be a nonzero scalar constant. Give an example of matrices A, C, and a constant vector x_o such that the linear system model with matrices A and C and initial state x_o yields the output $y(t) = c$ for all $t \geq 0$.

Solution: One choice is to let the state space be $\mathbb{R}$ (i.e. $n = 1$) and take A to be the scalar constant 0, let C be the 1×1 identity matrix (i.e. $C = 1$) and let the initial state be given by $x_o = c$. Then $x(t) = y(t) = c$ for all $t \geq 0$.

(b) Is there a possible answer to part (a) above such that the matrix A is full rank?

Solution: (Other solutions may be possible.) Consider any LTI system model with B and D equal to matrices of all zeros such that $y(t) = c$ for all $t \geq 0$. Then $Ce^{At}x_0 = c$ for all $t \geq 0$. Differentiating each side k times yields $CA^k e^{At}x_0 = \vartheta$ for all $k \geq 1$. Setting $t = 0$ yields $CA^k x_0 = \vartheta$ for all $k \geq 1$. For the sake of argument by contradiction, suppose that A has full rank. By the Cayley-Hamilton theorem, $\triangle(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1}A + \alpha_n I$. Furthermore, $\alpha_n = (-1)^n \det A \neq 0$, due to the assumption that A has full rank. Therefore I can be written as a linear combination of $A, A^2, \ldots, A^n$ so it follows that $Cx_0 = 0$. This contradicts the requirement that $y(0) = Cx_0 = c \neq 0$. Thus, the answer is no.

3. **[On the nonuniqueness of the fundamental matrix for LTV systems]**
Consider an LTV system defined by $\dot{x} = A(t)x$. Recall that the fundamental matrix $(U(t) : -\infty < t < \infty)$ is given by $U(t) = [\psi^1(t) \cdots \psi^n(t)]$ where $\psi^1, \ldots, \psi^n$ are linearly independent (as vectors in $C^n(-\infty, \infty)$, the space of continuous, $\mathbb{R}^n$-valued functions on $(-\infty, \infty)$) solutions of $\dot{x} = A(t)x$ for different values of $x(t_0)$ for some time t_0. Suppose U is one choice of fundamental matrix and $\overline{U}(t) = [\overline{\psi}^1(t) \cdots \overline{\psi}^n(t)]$ is another choice.

   (a) Explain why there is a nonsingular matrix P such that $\overline{U}(t) = U(t)P$ for all t.
   Solution: There are multiple ways to go about this. Here are two solutions. Solution 1: Since $\psi^1, \ldots, \psi^n$ is a basis for the vector space of all solutions to $\dot{x} = A(t)x$, for each k we can take the k^{th} column of P to be the vector of coordinates of $\overline{\psi}^k$ with respect to the basis $\psi^1, \ldots, \psi^n$. In other words, $p_{1,k}\psi_1 + \cdots + p_{n,k}\psi_n = \overline{\psi}_k$. The matrix P must be nonsingular because $\overline{\psi}^1, \ldots, \overline{\psi}^n$ is also a basis.
   Solution 2: For $\psi^1, \ldots, \psi^n$ to be linearly independent solutions, the columns of $U(t)$ must be linearly independent for all t (or else for every t they would not be linearly independent.) The same for $\overline{U}$. So we can let $P = U^{-1}(t_0)\overline{U}(t_0)$ for a fixed value of t_0. Then if we define $D(t) = \overline{U}(t) - U(t)P$ we have $D(t_0) = 0_{n\times n}$ and D satisfies the differential equation $\dot{D} = A(t)D$ which by uniqueness of solutions to the linear differential equation implies $D(t) = 0$ for all t. Thus, $\overline{U}(t) = U(t)P$ for all t.

   (b) Recall that the state transition matrix is given by $\phi(t, \tau) = U(t)U^{-1}(\tau)$. Does the state transition matrix depend on the choice of U? Justify your answer.
   Solution: Using $\overline{U}$ instead of U would give

$$\overline{U}(t)\overline{U}^{-1}(\tau) = U(t)P\left(P^{-1}U^{-1}(\tau)\right) = U(t)U^{-1}(\tau).$$

   Thus, ϕ does not depend on the choice of U.

4. **[A linear system with speed scaling]**
An example of linear time-varying system is given by $\dot{x} = A(t)x$, where $A(t) = s(t)A$ for all $-\infty < t < \infty$, where s is a piecewise continuous nonnegative function and A is a fixed matrix. An interpretation is that $s(t)$ is the speed of the system at time t. Note that if s is a constant function then the system is time invariant.

   (a) Let $(U(t) : -\infty < t < \infty)$ be the specific choice of fundamental matrix for the system such that $U(0) = I$. Show that $U(t) = e^{A\tau(t)}$, where $\tau(t) = \int_0^t s(u)du$ and give an intuitive explanation of this expression.
   Solution: We prove that $U = \overline{U}$ where $\overline{U}(t) = e^{A\tau(t)}$. It suffices to note that $\overline{U}(0) = I$ and then to check that $\dot{\overline{U}}(t) = A(t)\overline{U}(t)$. By the chain rule of calculus, $\dot{\overline{U}}(t) = \frac{\partial e^{A\tau}}{\partial \tau}\Big|_{\tau=\tau(t)} \frac{d\tau(t)}{dt} = Ae^{A\tau(t)}s(t) = A(t)\overline{U}(t)$, as desired. Intuitively, U traces out the same trajectory as the LTI system for matrix A, but at varying speed. Since $\tau(t)$ is the integral of speed up to time t, it indicates how far the system has traveled along its trajectory by time t. This is easier to understand for $t \geq 0$, but the intuition is the same for $t \leq 0$ going backwards in time.

   (b) Find a similar expression for the state transition matrix $(\phi(u, v) : u, v \in (-\infty, \infty))$.
   Solution: Using the facts $U^{-1}(v) = e^{-A\tau(v)}$ and $e^{aA}e^{bA} = e^{(a+b)A}$ for real values of a and b, we have $\phi(u, v) = U(u)U^{-1}(v) = e^{A(\tau(u)-\tau(v))} = e^{A\int_v^u s(t)dt}$. (Note that the

solutions in this problem don't require s to be nonnegative. If s can take both positive and negative values then the time varying system can run both forwards and backwards along the trajectory of the time invariant system for matrix A.)