University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 3
Solutions of State Equations

Due: Wednesday, September 18, 11:59 pm
Reading: Course notes, Chapter 3

1. **[Computation of state transition matrix for an LTI system]**
The goal of this problem is for you to see how to compute e^{At} by hand for

$$A = \begin{bmatrix} -1 & 4 & 6 \\ 0 & -2 & 5 \\ 0 & 0 & -3 \end{bmatrix} \tag{1}$$

three different ways.

   (a) Compute e^{At} by first finding the modal matrix and diagonalizing A. Work to completion.

   (b) Compute e^{At} by finding the Laplace transform of e^{At} and converting to the time domain. Carry out enough details to make sure you get the same answer as in part (a).

   (c) Show how e^{At} could be computed in this case by solving differential equations in the time domain to find the columns of e^{At}. This is possible due to the triangular structure of A. You do not need to carry out all details.

2. **[Constant output linear system model]**
Consider a linear time invariant (LTI) system with zero input – so the matrices B and D are irrelevant.

   (a) Let c be a nonzero scalar constant. Give an example of matrices A, C, and a constant vector x_o such that the linear system model with matrices A and C and initial state x_o yields the output $y(t) = c$ for all $t \geq 0$.

   (b) Is there a possible answer to part (a) above such that the matrix A is full rank?

3. **[On the nonuniqueness of the fundamental matrix for LTV systems]**
Consider an LTV system defined by $\dot{x} = A(t)x$. Recall that the fundamental matrix $(U(t) : -\infty < t < \infty)$ is given by $U(t) = [\psi^1(t) \cdots \psi^n(t)]$ where $\psi^1, \ldots, \psi^n$ are linearly independent (as vectors in $C^n(-\infty, \infty)$, the space of continuous, $\mathbb{R}^n$-valued functions on $(-\infty, \infty)$) solutions of $\dot{x} = A(t)x$ for different values of $x(t_0)$ for some time t_0. Suppose U is one choice of fundamental matrix and $\overline{U}(t) = [\overline{\psi}^1(t) \cdots \overline{\psi}^n(t)]$ is another choice.

   (a) Explain why there is a nonsingular matrix P such that $\overline{U}(t) = U(t)P$ for all t.

   (b) Recall that the state transition matrix is given by $\phi(t, \tau) = U(t)U^{-1}(\tau)$. Does the state transition matrix depend on the choice of U? Justify your answer.

4. **[A linear system with speed scaling]**
An example of linear time-varying system is given by $\dot{x} = A(t)x$, where $A(t) = s(t)A$ for all $-\infty < t < \infty$, where s is a piecewise continuous nonnegative function and A is a fixed matrix. An interpretation is that $s(t)$ is the speed of the system at time t. Note that if s is a constant function then the system is time invariant.

(a) Let $(U(t) : -\infty < t < \infty)$ be the specific choice of fundamental matrix for the system such that $U(0) = I$. Show that $U(t) = e^{A\tau(t)}$, where $\tau(t) = \int_0^t s(u)du$ and give an intuitive explanation of this expression.

(b) Find a similar expression for the state transition matrix $(\phi(u,v) : u, v \in (-\infty, \infty))$.