University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 2: Problems and Solutions
Vector Spaces

Due: Wednesday, September 11, 11:59pm
Reading: Course notes, Chapter 2

1. [**Elementary row operations and Gaussian elimination**]
   Consider the following system of linear equations for unknowns x, y, z :

$$\begin{aligned} 2x + 3y + z &= 5 \\ 4x + 3y + 2z &= 4 \\ x + y + z &= 12 \end{aligned}$$

   Solve for x, y, z using the method of Gaussian elimination. Show the augmented matrix at each step as in https://en.wikipedia.org/wiki/Gaussian_elimination.

   Solution: The sequence of augmented matrices is as follows:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 5 \\ 4 & 3 & 2 & 4 \\ 1 & 1 & 1 & 12 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 5 \\ 0 & -3 & 0 & -6 \\ 0 & -0.5 & 0.5 & 9.5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 5 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0.5 & 10.5 \end{array}\right]$$

   From which we sequentially find $z = 21$, $y = 2$, and $x = -11$.

2. [**Equality of dimensions of row and column span of a matrix**]
   Consider the following two elementary row operations on a matrix: swapping two rows or adding a constant times one row to another. Such operations can transform any $m \times n$ matrix into row echelon form. See https://en.wikipedia.org/wiki/Gaussian_elimination. For example, suppose a 5×8 matrix A can be reduced to the following matrix T by such operations:

$$T = \begin{bmatrix} a & * & * & * & * & * & * & * \\ 0 & 0 & b & * & * & * & * & * \\ 0 & 0 & 0 & c & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

   such that a, b, c are nonzero constants and each asterisk represents an arbitrary number.

   (a) What is a basis for the space spanned by the rows of T and what is the dimension of the space? Explain.

   Solution: The first three rows are linearly independent because if there were constants a_1, a_2, a_3 such that $a_1 r^1 + a_2 r^2 + a_3 r^3 = 0$ then $a_1 a = 0$ so $a_1 = 0$, and then $a_2 b = 0$ so

$a_2 = 0$, and then $a_3 c = 0$ so $a_3 = 0$. The fourth and fifth rows are trivially in the linear span of the first three rows. So the first three rows form a basis for the space spanned by the rows of T and the dimension of the row space is three.

(b) What is a basis for the space spanned by the columns of T and what is the dimension of the space? Explain.

Solution: Similarly the first, third, and fourth columns of T are linearly independent and all other columns are in the linear span of those three vectors. So those three columns form a basis for the column span of T which has dimensions three.

(c) Are the row spans of A and T necessarily the same? Are the dimensions of the spaces spanned by the rows of A and T, respectively, the same? Explain.

Solution: Yes, swapping two vectors or adding a constant times one vector to another does not change the span of the vectors. Since the spaces spanned by the rows of A and T are the same, those spaces have the same dimension.

(d) Are the column spans of A and T necessarily the same? Are the dimensions of the spaces spanned by the columns of A and T, respectively, the same? Explain.

Solution: No, the column spans of A and T don't have to be the same. For example, A could have the same first three rows as T and the last two rows of A could be the same as the third row of T, and those two matrices have different column spans.

An elementary row operation induces a change of coordinates on the column vectors of a matrix, which can be represented by multiplication of those vectors by a nonsingular matrix P. For example, if c represents one of the columns and the first and second rows are swapped, then c is transformed to Pc where

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix},$$

and in this case $P = P^{-1}$. Or if α times the first row is added to the second row then c is transformed to Pc where

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \alpha & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and P^{-1} is the same as P with α replaced by $-\alpha$ corresponding to subtracting α times the first row from the second row. A change of coordinates does not change the dimension of a subspace, so the dimensions of the column spans of A and T are equal.

(e) Explain why the dimension of the row span is equal to the dimension of the column span for any $m \times n$ matrix.

Solution: The story is the same as this example for arbitrary matrices. The result is true for matrices in row echelon form, and any matrix can be transformed to row echelon form by elementary row operations which preserve the dimensions of the row span and the column span.

3. **[The determinant of a matrix]**
There are many equivalent ways to define determinants of matrices. Here we describe an axiomatic approach. Let $n \geq 1$ and consider the space of $n \times n$ matrices, with real or complex valued entries. Given such a matrix A let $c^1, \ldots, c^n$ denote the columns of the matrix. Then determinant, denoted by det, is the mapping from $n \times n$ matrices to scalar values such that:

Linearity in each column If $c^1, \ldots, c^n$ represent the columns of A and a, b are scalars and y, z are n-vectors then

$$\det(c^1, \ldots, c^{i-1}, ay + bz, c^{i+1}, \ldots, c^n) =$$
$$a \det(c^1, \ldots, c^{i-1}, y, c^{i+1}, \ldots, c^n) + b \det(c^1, \ldots, c^{i-1}, z, c^{i+1}, \ldots, c^n)$$

Zero if two columns are the same If $c^i = c^j$ for distinct columns i and j of A then $\det A = 0$.

Normalization $\det I = 1$ where I is the $n \times n$ identity matrix.

Using the above axioms, prove the following properties of det .

(a) Show that if B is obtained from A by adding a scalar multiple of one column of A to another column of A then $\det B = \det A$.

Solution: If B is obtained from A by adding ac^i to c^j we use the linearity property to get:

$$\det B = \det(c^1, \ldots, c^{j-1}, c^j + ac^i, c^{j+1}, \ldots, c^n) =$$
$$\det(c^1, \ldots, c^{j-1}, c^j, c^{j+1}, \ldots, c^n) + a \det(c^1, \ldots, c^{j-1}, c^i, c^{j+1}, \ldots, c^n) = \det A + 0.$$

The zero comes because the i^{th} and j^{th} columns are the same in the matrix indicated.

(b) Show that if B is obtained from A by swapping two columns of A then $\det B = -\det A$. (Hint: Use part (a) and the linearity property.)

Solution: Consider a sequence of matrices starting from A such that the i^{th} and j^{th} columns are the only ones modified and are as shown:

c^i	c^j	matrix$A$
c^i	$c^i + c^j$	added i^{th} column to j^{th}
$-c^j$	$c^i + c^j$	subtracted j^{th} column from i^{th}
$-c^j$	c^i	added i^{th} column to j^{th}
c^j	c^i	multiplied i^{th} column by -1 to obtain matrix B

The only step that changes det is the one such that the i^{th} column is multiplied by -1, so that $\det B = -\det A$.

(c) Show that if there exists a mapping det satisfying the axioms then it is unique. Hint: The value of $\det A$ is uniquely determined by the axioms if A has a column of all zeros (by the linearity property) or if A is a diagonal matrix (by the linearity in each column and normalization axioms. So it suffices to show that any matrix A can be reduced to one of those matrices by using the elementary column operations of the following type: swapping two columns or adding a scalar multiple of one column to another column because the effects of each of those operations on the determinant are determined by the axioms.

Solution: Such elementary column operations can be used to transform A to reduced column echelon form A'. If one of the columns of A' is the zero column ϑ then $\det A' = \det A = 0$. Otherwise A' is a lower triangular matrix with nonzero diagonal elements so additional elementary column operation can reduce it to a diagonal matrix.

(d) Let $\widetilde{\det} A = \sum_\pi \mathsf{sgn}(\pi) \prod_{j=1}^n c^j_{\pi(j)}$, where the sum is over all $n!$ permutations π and $\mathsf{sgn}(\pi) = 1$ if π can be written as a composition of an even number of transpositions and $\mathsf{sgn}(\pi) = -1$ if π can be written as a composition of an odd number of transpositions. Show that $\widetilde{\det}$ satisfies the three axioms given in the problem statement. (This shows the existence of det and $\det = \widetilde{\det}$.)

Solution: Linearity in each column Let c^j_i denote the i^{th} element of the j^{th} column. Then

$$
\begin{aligned}
&\widetilde{\det}(c^1, ..., c^{i-1}, ay + bz, c^{i+1}, ..., c^n) \\
&= \sum_\pi \mathsf{sgn}(\pi) \left[\prod_{j=1, j\neq i}^n c^j_{\pi(j)} \right] (ay_{\pi(i)} + bz_{\pi(i)}) \\
&= \sum_\pi \mathsf{sgn}(\pi) \left[\prod_{j=1, j\neq i}^n c^j_{\pi(j)} \right] ay_{\pi(i)} + \sum_\pi \mathsf{sgn}(\pi) \left[\prod_{j=1, j\neq i}^n c^j_{\pi(j)} \right] bz_{\pi(i)} \\
&= a\widetilde{\det}(c^1, \ldots, c^{i-1}, y, c^{i+1}, \ldots, c^n) + b\widetilde{\det}(c^1, \ldots, c^{i-1}, z, c^{i+1}, \ldots, c^n)
\end{aligned}
$$

Zero if two columns are the same. If columns j and j' are the same, then the terms in the sum defining $\widetilde{\det}$ can be grouped into pairs $\{\pi, \pi'\}$ such that π' is obtained from π by transposing $\pi(j)$ and $\pi(j')$. The products for the two permutations in each pair sum to zero because $\mathsf{sgn}(\pi) = -\mathsf{sgn}(\pi')$, so the total sum defining $\widetilde{\det}$ is zero.

Normalization The only nonzero term in the sum defining $\widetilde{\det}(I)$ is for the identity permutation yielding a product equal to one so $\widetilde{\det}(I) = 1$.

(e) A very important and useful property of det for square matrices A with real values is that the absolute value, $|\det(A)|$, is the volume of the parallelepiped generated by the columns of A. Prove this property. (Hint: Say a matrix A satisfies property V if $|\det(A)|$ is the volume of the parallelepiped generated by the columns of A. Using the axioms and arguments from part (c) above show that all matrices have property V.) (Note: This property is nicely illustrated in the utube video *The essence of linear algebra Chapter 6* https://www.youtube.com/watch?v=Ip3X9LOh2dk).

Solution: By the result of part (c) it is sufficient to note that any matrix with a zero column or any diagonal matrix trivially satisfies property V and if A has property V and B is obtained from B by either of the two types of elementary column operations: (i) swapping two columns or (ii) adding a constant times one column to another column, then B has property V. (i) Swapping two columns changes neither the parallelepiped generated by the columns of A nor $|\det A|$ so if A has property V than so does any matrix obtained by swapping columns of A. (ii) If B is obtained from A by changing c^j to $c^j + ac^i$ then $\det B = \det A$. So it remains to argue that the volume of the parallelepipeds generated by the column vectors of A and B have the same volumes. For either A or B, the volume is equal to the $n-1$-dimensional volume of the base parallelepiped generated by $\{c^k : k \neq j\}$ times the distance between the hyperplane spanned by $\{c^k : k \neq j\}$ and the affine hyperplane obtained by displacing the first hyperplane by the vector in the

j^{th} column. Since c^i is in the base hyperplane, whether that hyperplane is displaced by c^j alone or by $c^j + ac^i$, it gives the same affine hyperplane, and hence the same volume.

4. **[Orthogonal subspaces]**
Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces over the complex field $\mathbb{C}$, each equipped with an inner product.

 (a) Given a subspace $\mathcal{V}$ of $\mathcal{Y}$, let $\mathcal{V}^{\perp}$ denote the set of all vectors in $\mathcal{Y}$ that are orthogonal to all vectors in $\mathcal{V}$. Show that $\mathcal{V}^{\perp}$ is a subspace of $\mathcal{Y}$.

 Solution: By its definition, $\mathcal{V}^{\perp}$ is a subset of $\mathcal{Y}$. It remains to show that $\mathcal{V}^{\perp}$ itself is a vector space. As noted in class, most of the required properties follow simply because $\mathcal{V}^{\perp}$ is a subset of $\mathcal{Y}$. It therefore suffices to show that for any $y_1, y_2 \in \mathcal{V}^{\perp}$ and any $\alpha_1, \alpha_2 \in \mathbb{C}$, that $\alpha_1 y_1 + \alpha_2 y_2 \in \mathcal{V}^{\perp}$. To prove it, if the given conditions hold then for all $v \in V$, $\langle v, \alpha_1 y_1 + \alpha_2 y_2 \rangle = \alpha_1 \langle v, y_1 \rangle + \alpha_2 \langle v, y_2 \rangle = 0 + 0 = 0$, so that $\alpha_1 y_1 + \alpha_2 y_2 \in \mathcal{V}^{\perp}$.

 (b) Suppose $\mathcal{A}$ is a linear mapping from $\mathcal{X}$ to $\mathcal{Y}$ with an adjoint mapping $\mathcal{A}^*$. Show that $\mathcal{N}(\mathcal{A}^*) = \mathcal{R}^{\perp}(\mathcal{A})$.

 Solution: Given any $y \in \mathcal{Y}$, the following statements are equivalent because each statement is equivalent to the one just above or below it:

 $y \in \mathcal{R}^{\perp}(\mathcal{A})$
 $\langle \mathcal{A}(x), y \rangle = 0$ for all $x \in \mathcal{X}$
 $\langle x, \mathcal{A}^*(y) \rangle = 0$ for all $x \in \mathcal{X}$
 $\mathcal{A}^*(y) = 0$
 $y \in \mathcal{N}(\mathcal{A}^*)$.

5. **[Eigenvalues and the trace of a square matrix A]**
The trace of a square matrix F, denoted by $\mathsf{Tr}(F)$, is the sum of the diagonal elements of F.

 (a) Let m, n be positive integers. The collection of $m \times n$ matrices with complex elements is a vector space over $\mathbb{C}$, which we denote by $\mathcal{M}_{m,n}(\mathbb{C})$. Given $A, B \in \mathcal{M}_{m,n}(\mathbb{C})$, let $\langle A, B \rangle = \mathsf{Tr}(A^* B)$, where A^* is the complex conjugate transpose of A. Show that $\langle ., . \rangle$ is an inner product for $\mathcal{M}_{m,n}(\mathbb{C})$ over $\mathbb{C}$. The corresponding norm induced on the elements of $\mathcal{M}_{m,n}(\mathbb{C})$ is called the Frobenius norm. How are $\langle ., . \rangle$ and the Frobenius norm related to the usual Euclidean inner product and vector norm?

 Solution: For $1 \leq i \leq n$, $(A^*B)_{i,i} = \sum_j \overline{a_{j,i}} b_{j,i}$. Therefore, $\langle A, B \rangle = \sum_i \sum_j \overline{a_{j,i}} b_{j,i}$. In words, $\langle A, B \rangle$ is the Euclidean inner product if we identify a matrix in $\mathcal{M}_{m,n}(\mathbb{C})$ with an mn-dimensional vector in $\mathbb{C}^{mn}$ by vertically stacking up the columns. From this it is easy to verify the three properties defining an inner product, namely $\langle A, B \rangle = \overline{\langle B, A \rangle}$, linearity of the mapping $A \mapsto \langle A, B \rangle$ and $\langle A, A \rangle \geq 0$ with equality if and only if A is the all zero matrix Z. Alternatively, we could just note that the usual Euclidean inner product is known to be an inner product. The Frobenius norm of a matrix A is given by $\sqrt{\langle A, A \rangle} = \sqrt{\sum_i \sum_j |a_{i,j}|^2}$.

 Altenative solution We prove that $\langle \cdot, \cdot \rangle$ as defined satisfies the axioms of a norm. Again we use the fact $\langle A, B \rangle = \sum_i \sum_j \overline{a_{j,i}} b_{j,i}$.

$$\begin{aligned}\langle A, \alpha B + \beta C\rangle &= \sum_i \sum_j \overline{a_{j,i}}(\alpha b_{j,i} + \beta c_{j,i}) \\ &= \sum_i \sum_j \alpha \overline{a_{j,i}} b_{j,i} + \beta \sum_i \sum_j \overline{a_{j,i}} c_{j,i} \\ &= \alpha\langle A, B\rangle + \beta\langle A, C\rangle\end{aligned}$$

$\langle A, B\rangle = \sum_i \sum_j \overline{a_{j,i}} b_{j,i} = \overline{\sum_i \sum_j a_{j,i} \overline{b_{j,i}}} = \overline{\langle B, A\rangle}$

$\langle A, A\rangle = \sum_i \sum_j \overline{a_{j,i}} a_{j,i} = \sum_i \sum_j |a_{j,i}|^2 \geq 0$, with equality if and only if A is the all zero matrix.

(b) Show that if A is an $m \times n$ matrix and B is an $n \times m$ matrix then $\mathsf{Tr}(AB) = \mathsf{Tr}(BA)$. (Hint: Solve directly – this is not a continuation of part (a).)

Solution: Both are equal to $\sum_i \sum_j a_{i,j} b_{j,i}$.

(c) Use the fact that A is similar to its Jordon canonical form J (i.e. there is a nonsingular matrix P such that $PAP^{-1} = J$) to show that $\mathsf{Tr}(A)$ is equal to the sum of the eigenvalues of A.

Solution: The diagonal elements of J are the eigenvalues of A, so $\mathsf{Tr}(J)$ is the sum of the eigenvalues of A. Furthermore, by part (b) above, $\mathsf{Tr}(J) = \mathsf{Tr}(PAP^{-1}) = \mathsf{Tr}(AP^{-1}P) = \mathsf{Tr}(AI) = \mathsf{Tr}(A)$.

(d) Using the fundamental theorem of algebra, identify the coefficient of λ^{n-1} in the characteristic polynomial $\triangle(\lambda)$ of A and then use properties of det and the definition of $\triangle(\lambda)$ to again show that $\mathsf{Tr}(A)$ is equal to the sum of the eigenvalues of A.

Solution: On one hand, by the fundamental theorem of algebra, $\triangle(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$ where $\lambda_1, \ldots, \lambda_n$ are the roots of the polynomial which are also the eigenvalues of A. Multiplying out the terms we see that the coefficient of λ^{n-1} is the negative sum of the eigenvalues.

On the other hand, by definition we have $\triangle(\lambda) = \det(I\lambda - A)$. Using the formula for det involving a sum over permutations, we can see that the terms in $\det(I\lambda - A)$ that are scalars times λ^{n-1} must come from the identity permutation – all other permutations would miss at least two diagonal elements and involve smaller powers of λ. So each term with λ^{n-1} comes from the product of λ in $n-1$ diagonal elements of $\lambda I - A$ times $-a_{i,i}$ for some i. Summing over i yields that the coefficient of λ^{n-1} in $\triangle(\lambda)$ is $-\mathsf{Tr}(A)$. Equating the two expressions for the coefficient of λ^{n-1} in $\triangle(\lambda)$ gives the desired result.