**University of Illinois** **Fall 2024**

# ECE 515/ME 540: Problem Set 2
## Vector Spaces

**Due:** Wednesday, September 11, 11:59pm
**Reading:** Course notes, Chapter 2

1. [**Elementary row operations and Gaussian elimination**]
   Consider the following system of linear equations for unknowns $x, y, z$ :

$$\begin{aligned} 2x + 3y + z &= 5 \\ 4x + 3y + 2z &= 4 \\ x + y + z &= 12 \end{aligned}$$

   Solve for $x, y, z$ using the method of Gaussian elimination. Show the augmented matrix at each step as in https://en.wikipedia.org/wiki/Gaussian_elimination.

2. [**Equality of dimensions of row and column span of a matrix**]
   Consider the following two elementary row operations on a matrix: swapping two rows or adding a constant times one row to another. Such operations can transform any $m \times n$ matrix into row echelon form. See https://en.wikipedia.org/wiki/Gaussian_elimination. For example, suppose a $5 \times 8$ matrix $A$ can be reduced to the following matrix $T$ by such operations:

$$T = \begin{bmatrix} a & * & * & * & * & * & * & * \\ 0 & 0 & b & * & * & * & * & * \\ 0 & 0 & 0 & c & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

   such that $a, b, c$ are nonzero constants and each asterisk represents an arbitrary number.

   (a) What is a basis for the space spanned by the rows of $T$ and what is the dimension of the space? Explain.

   (b) What is a basis for the space spanned by the columns of $T$ and what is the dimension of the space? Explain.

   (c) Are the row spans of $A$ and $T$ necessarily the same? Are the dimensions of the spaces spanned by the rows of $A$ and $T$, respectively, the same? Explain.

   (d) Are the column spans of $A$ and $T$ necessarily the same? Are the dimensions of the spaces spanned by the columns of $A$ and $T$, respectively, the same? Explain.

   (e) Explain why the dimension of the row span is equal to the dimension of the column span for any $m \times n$ matrix.

3. [**The determinant of a matrix**]
   There are many equivalent ways to define determinants of matrices. Here we describe an axiomatic approach. Let $n \geq 1$ and consider the space of $n \times n$ matrices, with real or complex valued entries. Given such a matrix $A$ let $c^1, \ldots, c^n$ denote the columns of the matrix. Then determinant, denoted by det, is the mapping from $n \times n$ matrices to scalar values such that:

**Linearity in each column** If $c^1, \ldots, c^n$ represent the columns of $A$ and $a, b$ are scalars and $y, z$ are $n$-vectors then

$$\det(c^1, \ldots, c^{i-1}, ay + bz, c^{i+1}, \ldots, c^n) =$$
$$a \det(c^1, \ldots, c^{i-1}, y, c^{i+1}, \ldots, c^n) + b \det(c^1, \ldots, c^{i-1}, z, c^{i+1}, \ldots, c^n)$$

**Zero if two columns are the same** If $c^i = c^j$ for distinct columns $i$ and $j$ of $A$ then $\det A = 0$.

**Normalization** $\det I = 1$ where $I$ is the $n \times n$ identity matrix.

Using the above axioms, prove the following properties of det.

(a) Show that if $B$ is obtained from $A$ by adding a scalar multiple of one column of $A$ to another column of $A$ then $\det B = \det A$.

(b) Show that if $B$ is obtained from $A$ by swapping two columns of $A$ then $\det B = -\det A$. (Hint: Use part (a) and the linearity property.)

(c) Show that if there exists a mapping det satisfying the axioms then it is unique. Hint: The value of $\det A$ is uniquely determined by the axioms if $A$ has a column of all zeros (by the linearity property) or if $A$ is a diagonal matrix (by the linearity in each column and normalization axioms. So it suffices to show that any matrix $A$ can be reduced to one of those matrices by using the elementary column operations of the following type: swapping two columns or adding a scalar multiple of one column to another column because the effects of each of those operations on the determinant are determined by the axioms.

(d) Let $\widetilde{\det} A = \sum_\pi \mathsf{sgn}(\pi) \prod_{j=1}^n c^j_{\pi(j)}$, where the sum is over all $n!$ permutations $\pi$ and $\mathsf{sgn}(\pi) = 1$ if $\pi$ can be written as a composition of an even number of transpositions and $\mathsf{sgn}(\pi) = -1$ if $\pi$ can be written as a composition of an odd number of transpositions. Show that $\widetilde{\det}$ satisfies the three axioms given in the problem statement. (This shows the existence of det and $\det = \widetilde{\det}$.)

(e) A very important and useful property of det for square matrices $A$ with real values is that the absolute value, $|\det(A)|$, is the volume of the parallelepiped generated by the columns of $A$. Prove this property. (Hint: Say a matrix $A$ satisfies property $V$ if $|\det(A)|$ is the volume of the parallelepiped generated by the columns of $A$. Using the axioms and arguments from part (c) above show that all matrices have property $V$.) (Note: This property is nicely illustrated in the utube video *The essence of linear algebra Chapter 6* https://www.youtube.com/watch?v=Ip3X9LOh2dk).

4. **[Orthogonal subspaces]**
Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces over the complex field $\mathbb{C}$, each equipped with an inner product.

(a) Given a subspace $\mathcal{V}$ of $\mathcal{Y}$, let $\mathcal{V}^\perp$ denote the set of all vectors in $\mathcal{Y}$ that are orthogonal to all vectors in $\mathcal{V}$. Show that $\mathcal{V}^\perp$ is a subspace of $\mathcal{Y}$.

(b) Suppose $\mathcal{A}$ is a linear operator from $\mathcal{X}$ to $\mathcal{Y}$ with an adjoint operator $\mathcal{A}^*$. Show that $\mathcal{N}(\mathcal{A}^*) = \mathcal{R}^\perp(\mathcal{A})$.

5. **[Eigenvalues and the trace of a square matrix $A$]**
The trace of a square matrix $F$, denoted by $\mathsf{Tr}(F)$, is the sum of the diagonal elements of $F$.

(a) Let $m, n$ be positive integers. The collection of $m \times n$ matrices with complex elements is a vector space over $\mathbb{C}$, which we denote by $\mathcal{M}_{m,n}(\mathbb{C})$. Given $A, B \in \mathcal{M}_{m,n}(\mathbb{C})$, let $\langle A, B \rangle = \mathsf{Tr}(A^*B)$, where $A^*$ is the complex conjugate transpose of $A$. Show that $\langle ., . \rangle$ is an inner product for $\mathcal{M}_{m,n}(\mathbb{C})$ over $\mathbb{C}$. The corresponding norm induced on the elements of $\mathcal{M}_{m,n}(\mathbb{C})$ is called the Frobenius norm. How are $\langle ., . \rangle$ and the Frobenius norm related to the usual Euclidean inner product and vector norm?

(b) Show that if $A$ is an $m \times n$ matrix and $B$ is an $n \times m$ matrix then $\mathsf{Tr}(AB) = \mathsf{Tr}(BA)$. (Hint: Solve directly – this is not a continuation of part (a).)

(c) Use the fact that $A$ is similar to its Jordon canonical form $J$ (i.e. there is a nonsingular matrix $P$ such that $PAP^{-1} = J$) to show that $\mathsf{Tr}(A)$ is equal to the sum of the eigenvalues of $A$.

(d) Using the fundamental theorem of algebra, identify the coefficient of $\lambda^{n-1}$ in the characteristic polynomial $\triangle(\lambda)$ of $A$ and then use properties det and the definition of $\triangle(\lambda)$ to again show that $\mathsf{Tr}(A)$ is equal to the sum of the eigenvalues of $A$.