University of Illinois **Fall 2024**

ECE 515/ME 540: Problem Set 1: Problems and Solutions
State Space models

Due: Wednesday, September 4, 11:59pm
Reading: Course notes, Chapter 1

1. [**State space realizations for a transfer function**]
Consider the SISO LTI system with the transfer function

$$G(s) = \frac{s+1}{(s+2)(s+3)(s+4)}.$$

(a) Obtain a state space model of the system in the controllable canonical form
Solution: Multiplying out gives

$$G(s) = \frac{s+1}{s^3 + 9s^2 + 26s + 24}$$

so the CCF is given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x \end{aligned}$$

(b) Now obtain a state space representation in the observable canonical form.
Solution: The OCF is given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \end{aligned}$$

(c) Using partial fraction expansion to obtain a representation with a diagonal state matrix A (modal form).
Solution: By partial fraction expansion we find:

$$G(s) = \frac{-0.5}{s+2} + \frac{2}{s+3} + \frac{-1.5}{s+4}$$

So the modal form is give by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} -0.5 \\ 2 \\ -1.5 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x \end{aligned}$$

2. **[State space realizations for a transfer function (version 2)]**
Repeat the previous problem for the SISO LTI system with the transfer function

$$G(s) = \frac{2s^3+1}{(s+2)(s+3)(s+4)}.$$

Solution: (a) Multiplying out and separating out the DC term yields:

$$\begin{aligned} G(s) &= \frac{2s^3+1}{s^3+9s^2+26s+24} \\ &= 2 - \frac{2s^3+18s^2+52s+48}{s^3+9s^2+26s+24} + \frac{2s^3+1}{s^3+9s^2+26s+24} \\ &= 2 + \frac{-18s^2-52s-47}{s^3+9s^2+26s+24} \end{aligned}$$

so the CCF is given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -47 & -52 & -18 \end{bmatrix} x + [2]\, u \end{aligned}$$

Solution: (b) The OCF is given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -18 \\ -52 \\ -47 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + [2]\, u \end{aligned}$$

Solution: (c) By partial fraction expansion we find:

$$G(s) = 2 + \frac{-7.5}{s+2} + \frac{53}{s+3} + \frac{-63.5}{s+4}$$

So the modal form is given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} -7.5 \\ 53 \\ -63.5 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x + [2]\, u \end{aligned}$$

3. **[Linearization about an equilibrium]**
Consider the nonlinear system with the state equation:

$$\ddot{y} + \dot{y} + y^2 = u^3$$

(a) Find the linear system obtained by linearizing about the equilibrium $y_e = u_e = 1$
Solution: Using the fact the derivative of u^3 is $3u^2$ which is equal to 3 at $u_u = 1$,

$$\ddot{\delta y} + \dot{\delta y} + (1 + 2\delta y) = 1 + 3\delta u$$

which simplifies to:

$$\ddot{\delta y} + \dot{\delta y} + 2\delta y = 3\delta u$$

(b) Express the linearized system in state space form. To be definite, give the canonical controllable form of the state space system.

Solution: Letting $x_1 = y$ and $x_2 = \dot{y}$ we have

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u$$
$$\delta y = \begin{bmatrix} 3 & 0 \end{bmatrix} \delta x$$

(Note: Instead of putting $3\delta u$ in the state equation we put the 3 in the output equation to conform with controllable canonical form.)

4. **[Pole placement of system in controllable canonical form by state feedback]** Consider the following state space model in controllable canonical form:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

(a) What is the characteristic polynomial and what are its roots? (As we will see somewhat later in the course, the characteristic polynomial is the denominator of the transfer function and its roots are the poles of the system.) Hint: One pole is -1.

Solution: $\triangle(s) = s^3 + s^2 - 4s - 4 = (s+1)(s+2)(s-2)$ which has roots -1, -2, 2

(b) Let $u = -Kx + r$ where

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}.$$

We interpret r as the new control input and the system with input r and output y now has state feedback. Give the new state space model with input r and output y. It should again be in controllable canonical form.

Solution: Substituting in the expression for u gives:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (-Kx + r)$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

or

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 - k_1 & 4 - k_2 & -1 - k_3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

(c) For what choice of K are the roots of the new system $-1, -2, -2$?

Solution: We would like the new characteristic polynomial to be $\bar{\triangle}(s) = (s+1)(s+2)^2 = s^3 + 5s^2 + 8s + 4$ so that the system model with feedback incorporated should be

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

Therefore, comparing with the solution to part (b), we take

$$K = \begin{bmatrix} 8 & 12 & 4 \end{bmatrix}$$