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**ECE 515 / ME 540** **Fall 2024 Final Exam. Dec. 17, 2024**

**3 hours; 3 pages of notes 2-sided OK; closed book; no calculators**

1. [**Problem 1 (16 points)**]
Consider the following SISO LTI system:

$$\dot{x} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x.$$

(a) Is the system controllable? Justify your answer.

**Solution:** Controllable because, for example, $\mathcal{C} = \begin{bmatrix} 1 & 3 & 9 \\ 0 & 0 & 2 \\ 1 & 4 & 13 \end{bmatrix}$ is full rank.

(b) Is the system observable? Justify your answer.

**Solution:** No because, for example, the observability matrix is $\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix}$, which is not full rank. Also, the system is in Kalman observability canonical form (KOCF) with a two dimensional unobservable part.

(c) Is the system detectable? Justify your answer.

**Solution:** No, because the system is in KOCF form and the unobservable submatrix $A_{\bar{o}} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ has an unstable eigenvalue 1.

(d) Is the system BIBO stable? Justify your answer.

**Solution:** No. The system is in KOCF form, it is controllable, and the observable subsystem matrix $A_o = [3]$ is unstable. A related way to see it is to note that $\dot{x_1} = 3x_1 + u$ so a bounded control such as $u = 1$ for $t \geq 0$ can make $x_1$ unbounded.

2. [**Problem 2 (15 points)**]

Let $A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 2 \\ 1 & 6 & 1 \end{pmatrix}$.

(a) What is the dimension of the column span of $A$?

**Solution:** Two. The matrix $A$ has rank two as can be readily determined by reducing to column echelon or row echelon form. Or we could note that the first two columns are linearly independent by not all three of them are – see part (c).

(b) What is the dimension of the row span of $A$?

**Solution:** Two.

(c) What is the null space of $A$?

**Solution:** Solving $A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vartheta$ we find $\mathcal{N}(A) = \text{span}\left\{\begin{bmatrix} 8 \\ -1 \\ -2 \end{bmatrix}\right\}$.

3. **[Problem 3 (10 points)]**

Let $A = \begin{bmatrix} 0 & 5 \\ -5 & -8 \end{bmatrix}$

(a) Find the stable subspace for the dynamics $\dot{x} = Ax$.

**Solution:** The characteristic polynomial of $A$ is $\det(Is - A) = \det\begin{bmatrix} s & -5 \\ 5 & s+8 \end{bmatrix} = s^2 + 8s + 25$, so the eigenvalues of $A$ are $\frac{-8 \pm \sqrt{-36}}{2} = -4 \pm 3j$. Since both eigenvalues have negative real parts $A$ is a Hurwitz matrix and the stable subspace is all of $\mathbb{R}^2$.

(b) Find constants $a$ and $b$ so that $aA^2 + bA^3 = I_{2\times 2}$, where $I_{2\times 2}$ is the $2 \times 2$ identity matrix.

**Solution:** By the Cayley-Hamilton theorem $A^2 + 8A + 25I = 0$, so $I = -\frac{8}{25}A - \frac{1}{25}A^2$. It follows that $A = -\frac{8}{25}A^2 - \frac{1}{25}A^3$. Substituting the second identity into the first one yields

$$\begin{aligned} I &= -\frac{8}{25}\left[-\frac{8}{25}A^2 - \frac{1}{25}A^3\right] - \frac{1}{25}A^2 \\ &= \frac{64-25}{625}A^2 + \frac{8}{625}A^3 \\ &= \frac{39}{625}A^2 + \frac{8}{625}A^3 \end{aligned}$$

This can also be solved by brute force. First calculating

$$A^2 = \begin{bmatrix} -25 & -40 \\ 40 & 39 \end{bmatrix} \text{ and } A^3 = \begin{bmatrix} 200 & 195 \\ -195 & -112 \end{bmatrix}.$$

Then $40a - 195b = 0$ and $-25a + 200b = 1$. Solving yields the same solution. It is the Cayley-Hamilton theorem that implies that there is a solution – there are four equations but only two of them are independent.

4. **[Problem 4 (20 points)]**

Consider the LTI system and cost function:

$$\dot{x} = u \qquad x(0) = 0 \qquad V(u) = \int_0^T u^4(\tau)d\tau + (x(T) - b)^4$$

where $b$ and $T$ are known constants with $T > 0$. The goal is to find the optimal control by using the minimum principle with no additional assumptions.

(a) Find the differential equation for the costate $p$ including the boundary condition, and solve the equation (the solution should depend on $x(T)$ and $b$).

**Solution:** Note that $H(x, u, p, t) = u^4 + pu$ and $m(x) = (x - b)^4$. Since $-\nabla_x H = 0$ and $\nabla_x m(x) = 4(x - b)^3$ the differential equation for $p$ is

$$\dot{p} = 0 \qquad p(T) = 4(x(T) - b)^3 \quad \text{so } p(t) = 4(x(T) - b)^3 \quad \text{for } 0 \le t \le T.$$

(b) Express the optimal control $u = (u(t))_{0 \le t \le T}$ as a function of the costate $p$.

**Solution:** Solving $0 = \nabla_u H(x, v, p, t) = 4u^3 + p$ yields $u = -\left(\frac{p}{4}\right)^{1/3}$. Since $p$ is constant in time so is the control.

(c) Combine parts (a) and (b) to write the control as a function of $b$ and $x(T)$.

**Solution:** Since $\frac{p}{4} = (x(T) - b)^3$ from (b) we get $u = -(x(T) - b) = b - X(T)$.

(d) Identify an equation that can be used to solve for $x(T)$ in terms of $b$ and solve it to find $x(T)$ in terms of $b$ and $T$.

**Solution:** Since the initial state is 0 and the control $u$ is constant we have $x(T) = Tu$ or $x(T) = T(b - x(T))$. Solving for $x(T)$ gives $x(T) = \frac{bT}{1+T}$. [Note: This implies that the optimal control is $u = \frac{b}{1+T}$ and the minimum cost is $\frac{b^4}{(1+T)^3}$.]

5. **[Problem 5 (10 points)]**

Consider the system model and cost function:

$$\dot{x} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \qquad V(u) = \int_0^\infty \|x(t)\|^2 + u^2(t) dt$$

where the initial condition, $x_0$, is arbitrary.

(a) What condition on the constants $a, b, c, d, e$ is necessary and sufficient for the optimal cost, $\min_u V(u)$, to be finite?

**Solution:** This is an LQR problem. It is necessary that $(A, B)$ be stabilizable and $(A, C)$ be detectable, where $C = Q = I_{3\times3}$. The system is in KCCF form so a necessary and sufficient condition for stabilizability is that the submatrix $A_{\bar{c}} = \begin{bmatrix} b & c \\ d & e \end{bmatrix}$ be Hurwitz. Since $C$ has rank 3, $(A, C)$ is observable so no other condition is needed.

(b) Assuming the condition in part (a) is true, derive the optimal control in state feedback form. (It can depend on one or more of the constants $a, b, c, d, e$.)

**Solution:** The variable $x_1$ satisfies $\dot{x_1} = ax_1$ and $V(u) = V_1(u) + V_{2,3}$ where $V_1(u) = \int_0^\infty |x_1(t)|^2 + u^2(t)dt$ and $V_{1,2} = \int_0^\infty x_2^2(t) + x_3^2(t)dt$. So the optimal control is the same as the control that minimizes $V_1(u)$ for the dynamics $\dot{x}_1 = ax_1$. The ARE for this subsystem is $2ap + 1 - p^2 = 0$ which has solution $p = \sqrt{1 + a^2} + a$, so the optimal control in state feedback form is $u = -\left[\sqrt{1 + a^2} + a\right] x_1$.

Note: Alternatively we could solve the ARE for the entire third order system. However we can see that there is a solution with $\bar{P}$ being block diagonal, with a $1 \times 1$ and a $2 \times 2$ block on the diagonal. If we plug in such a form for $P$ the ARE decouples into one for each block. The equation for the $1 \times 1$ block is the same as above. The equation for the $2 \times 2$ block is the same as the Lyapyunov stability equation for the uncontrollable subsystem. Under the stabilizability and observability conditions, we know there is a unique positive definite solution to the ARE equation, and so the block diagonal solution must be it.

6. **[Problem 6 (12 points)]**

Consider the time reversed scalar Ricatti differential equation:

$$\dot{p} = 2ap + q - p^2,$$

where $a$ and $q$ are constants and $q > 0$.

(a) Express the nonnegative equilibrium point, $p_e$ (also known as $\bar{p}$) in terms of $a$ and $q$.

**Solution:** Setting $\dot{p} = 0$ yields $p_e = \sqrt{a^2 + q} + a$.

(b) Find the linear system dynamics about the equilibrium point and determine if you can deduce any stability property for $p_e$ for the nonlinear dynamics.

**Solution:** With $f(p) = 2ap + q - p^2$ we have $f'(p_e) = 2a - 2p_e = -2\sqrt{a^2 + q} < 0$. Thus, with $p = p_e + \delta p$, we have $\dot{p} = \dot{\delta p} = f(p_e + \delta p) = f(p_e) + f'(p_e)\delta p + o(\delta p)$. Ignoring the higher order term gives the linear system

$$\dot{\delta p} = -2\left(\sqrt{a^2 + q}\right)\delta p.$$

Since $-2\sqrt{a^2 + q} < 0$ we can conclude that $p_e$ is locally asymptotically stable for the nonlinear dynamics.

(c) Determine whether $(0, \infty)$ is a region of asymptotic stability for $p_e$ under the nonlinear dynamics. Justify your answer.

**Solution:** The graph of $f$ is a downward facing parabola such that $f(0) > 0$ and $f(p_e) = 0$. Therefore $f(p) > 0$ for $0 < p < p_e$ and $f(p) < 0$ for $p_e < p < \infty$. Thus, $p$ is globally asymptotically stable. [Note: The above observation implies that $V(p) = \frac{1}{2}(p - p_e)^2$ is a Lyapunov function. In fact any bowl shaped positive definite function over $(0, \infty)$ for the equilibrium point $p_e$ is a Lyapunov function, because $\frac{d}{dt}V(p(t)) = V'(p(t))f(p(t))$.]

7. **[Problem 7a (8 points)]**
The following are unrelated short answer questions. For each provide a justification of your answer for full credit. (There is a lot of space on this page and the next page but you don't need to fill them up! We just didn't want to leave a blank page on the exam to fit on gradescope!)

(a) Suppose SISO LTI systems $(A, B, C, D)$ and $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ have the same transfer function. Under what fairly general condition on the two systems can we conclude they are equivalent up to state space transformation?

**Solution:** It is sufficient for both systems to be minimal (equivalently, controllable and observable). Then each would be equivalent to the CCF for their transfer function and hence to each other under state space transformation.

(b) Suppose we have designed a static state feedback law $u = -Kx$ to stabilize an LTI system. And suppose we don't have access to the state but only have access to the output $y = Cx$. How might we obtain a stable system using output feedback (possibly dynamic output feedback)? Under what condition on $(A, C)$ do we expect this to work?

**Solution:** We could replace $u = -Kx$ by $u - K\hat{x}$ where $\hat{x}$ is produced by an observer: $\dot{\hat{x}} = (A - BK)\hat{x} + L(y - C\hat{x}$. If $(A, C)$ is detectable we can select $L$ to make $A - LC$ Hurwitz which will lead to convergence $x(t) - \hat{x}(t) \to 0$. It would be even better if $(A, C)$ were observable so we could place the poles of $L$ arbitrarily.

8. **[Problem 7b (4+5=9 points)]**
The following are unrelated short answer questions. For each provide a justification of your answer for full credit.

(a) What is an advantage the minimum principle has over the HJB method and what advantage does the HJB method have over the minimum principle?

**Solution:** The minimum principle does not require solving a PDE while the more complex HJB method identifies a globally optimal control.

(b) Given an LTI model with matrices $A, B, C, D$ as usual, consider the following three subspaces:

$\Sigma_s$ : the stable subspace

$\Sigma_c$: the controllable subspace

$\Sigma_{\bar{o}}$: the unobservable subspace

Which of these subspace(s) is invariant under the system evolution with arbitrary control inputs? (In other words, for which one(s) of these subspaces do we know that if $x(0)$ is in the subspace and $\dot{x} = Ax + Bu$ for all $t \geq 0$ then $x(t)$ is in the subspace for all $t \geq 0$ for any choice of the control $u$?)

**Solution:** Only $\Sigma_c$ is invariant under the system dynamics with arbitrary control. $\Sigma_c$ is the range of the controllability matrix $\mathcal{C}$ which is invariant under multiplication by $A$ and hence under multiplication by $e^{At}$. It also contains the range of $B$. Since $x(t)$ is a linear combination of $e^{At}x(0)$ and $e^{A(t-\tau)}u(\tau)d\tau$ it is in $\Sigma_c$ it if $x(t_0)$ is. If $B = I$ for example then the control can move the state arbitrarily so that neither $\Sigma_s$ (determined by $A$ alone) nor $\Sigma_{\bar{o}}$ (determined by $(A, C)$ alone) are invariant for arbitrary controls unless they happen to be the whole of $\mathbb{R}^n$.