Name: ____________________ **NetID:** __________

ECE 515 / ME 540 **Fall 2024 Final Exam. Dec. 17, 2024**

3 hours; 3 pages of notes 2-sided OK; closed book; no calculators

1. **[Problem 1 (16 points)]**
   Consider the following SISO LTI system:

$$\dot{x} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x.$$

   (a) Is the system controllable? Justify your answer.

   (b) Is the system observable? Justify your answer.

   (c) Is the system detectable? Justify your answer.

   (d) Is the system BIBO stable? Justify your answer.

2. **[Problem 2 (15 points)]**

Let $A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 2 \\ 1 & 6 & 1 \end{pmatrix}$.

(a) What is the dimension of the column span of A?

(b) What is the dimension of the row span of A?

(c) What is the null space of A?

3. **[Problem 3 (10 points)]**

Let $A = \begin{bmatrix} 0 & 5 \\ -5 & -8 \end{bmatrix}$

(a) Find the stable subspace for the dynamics $\dot{x} = Ax$.

(b) Find constants a and b so that $aA^2 + bA^3 = I_{2\times 2}$, where $I_{2\times 2}$ is the 2×2 identity matrix.

4. **[Problem 4 (20 points)]**
Consider the LTI system and cost function:

$$\dot{x} = u \qquad x(0) = 0 \qquad V(u) = \int_0^T u^4(\tau)d\tau + (x(T) - b)^4$$

where b and T are known constants with $T > 0$. The goal is to find the optimal control by using the minimum principle with no additional assumptions.

(a) Find the differential equation for the costate p including the boundary condition, and solve the equation (the solution should depend on $x(T)$ and b).

(b) Express the optimal control $u = (u(t))_{0 \le t \le T}$ as a function of the costate p.

(c) Combine parts (a) and (b) to write the control as a function of b and $x(T)$.

(d) Identify an equation that can be used to solve for $x(T)$ in terms of b and solve it to find $x(T)$ in terms of b.

5. **[Problem 5 (10 points)]**

Consider the system model and cost function:

$$\dot{x} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \qquad V(u) = \int_0^\infty \|x(t)\|^2 + u^2(t)dt$$

where the initial condition, x_0, is arbitrary.

(a) What condition on the constants a, b, c, d, e is necessary and sufficient for the optimal cost, $\min_u V(u)$, to be finite?

(b) Assuming the condition in part (a) is true, derive the optimal control in state feedback form. (It can depend on one or more of the constants a, b, c, d, e.)

6. **[Problem 6 (12 points)]**
Consider the time reversed scalar Ricatti differential equation:

$$\dot{p} = 2ap + q - p^2,$$

where a and q are constants and $q > 0$.

(a) Express the nonnegative equilibrium point, p_e (also known as $\bar{p}$) in terms of a and q.

(b) Find the linear system dynamics about the equilibrium point and determine if you can deduce any stability property for p_e for the nonlinear dynamics.

(c) Determine whether $(0, \infty)$ is a region of asymptotic stability for p_e under the nonlinear dynamics. Justify your answer.

7. **[Problem 7a (8 points)]**
The following are unrelated short answer questions. For each provide a justification of your answer for full credit. (There is a lot of space on this page and the next page but you don't need to fill them up! We just didn't want to leave a blank page on the exam to fit on gradescope!)

   (a) Suppose SISO LTI systems (A, B, C, D) and $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ have the same transfer function. Under what fairly general condition on the two systems can we conclude they are equivalent up to state space transformation?

   (b) Suppose we have designed a static state feedback law $u = -Kx$ to stabilize an LTI system. And suppose we don't have access to the state but only have access to the output $y = Cx$. How might we obtain a stable system using output feedback (possibly dynamic output feedback)? Under what condition on (A, C) do we expect this to work?

8. **[Problem 7b (4+5=9 points)]**
The following are unrelated short answer questions. For each provide a justification of your answer for full credit.

(a) What is an advantage the minimum principle has over the HJB method and what advantage does the HJB method have over the minimum principle?

(b) Given an LTI model with matrices A, B, C, D as usual, consider the following three subspaces:

Σ_s : the stable subspace
Σ_c: the controllable subspace
$\Sigma_{\bar{o}}$: the unobservable subspace

Which of these subspace(s) is invariant under the system evolution with arbitrary control inputs? (In other words, for which one(s) of these subspaces do we know that if $x(0)$ is in the subspace and $\dot{x} = Ax + Bu$ for all $t \geq 0$ then $x(t)$ is in the subspace for all $t \geq 0$ for any choice of the control u?)