Name: ____________________ **NetID:** __________

ECE 515 / ME 540 Fall 2024 Midterm 2. Nov. 12, 2024

75 min; 2 page notes 2-sided OK; closed book; no calculators

1. **[Problem 1 (32 points)]**
   Consider the following SISO LTI system:

$$\dot{x} = \begin{bmatrix} 0 & 4 & 5 \\ 2 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x.$$

   (a) Is the system controllable? Justify your answer.
   Solution: Not controllable. For example, we see $\dot{x}_3 = 3x_3$ so the control can't be used to steer x_3. We could also look at the controllability matrix or consider the PBH/Hautus Rosebrock test.

   (b) Is the system observable? Justify your answer.
   Solution: Yes. The observability matrix is $\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 5 \\ 8 & 8 & 39 \end{bmatrix}$, which is full rank so the system is observable.

   (c) Is the system stabilizable? Justify your answer.
   Solution: No, because the uncontrollable coordinate identified in part (a) converges to infinity no matter what control is applied if it is initially nonzero. We can also see that the system is in KCCF form with $A_{\bar{c}} = [3]$, which is not Hurwitz.

   (d) Is the system BIBO stable? Justify your answer.
   Solution: No. One way to see it is to note that the system is observable and in KCCF form, so it is stabilizable if and only if $A_c = \begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}$ is Hurwitz, which it is not becasue it has a positive eigenvalue 4. Another way to see it is if $u(t) = 1$ for all $t \geq 0$ then x_1 and x_2 will both be nonnegative and $x_1(t) \geq t$, so $y(t) \to \infty$. Or we could note that the impulse response function for the system, $g(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \exp\left(\begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix} t\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, does not have a finite integral over $[0, \infty)$.

2. **[Problem 2 (20 points)]**
   Consider the following transfer function for a MIMO LTI system.

$$G(s) = \begin{bmatrix} \frac{1}{s+1} + \frac{2}{s+2} + \frac{3}{s+3} & \frac{-2}{s+2} + \frac{6}{s+3} \\ \frac{-2}{s+2} + \frac{1}{s+3} & \frac{1}{s+1} + \frac{2}{s+2} + \frac{2}{s+3} \end{bmatrix}.$$

   (a) What is the smallest possible model order for an LTI system that realizes G? Show your work.

Solution: $n = 4$ because $G(s) = \frac{1}{s+1}R_1 + \frac{1}{s+2}R_2 + \frac{1}{s+3}R_3$ where $R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R_2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$, and $R_3 = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$ with ranks 2,1,1 summing to 4.

(b) Find matrices A, B and C of an LTI system that realizes G with the minimum model order.

Solution:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

More generally, for this choice of A we can select any B and C of the form $B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$ and $C = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}$ such that $C_iB_i = R_i$ with the appropriate dimensions.

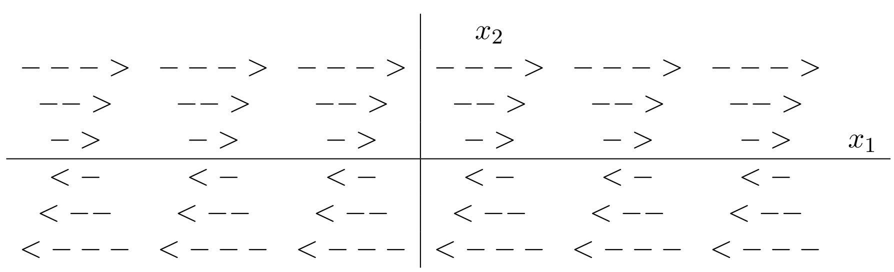
3. **[Problem 3 (24 points)]**

Consider the following SISO LTI system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x.$$

(a) Sketch the two dimensional phase plot/vector field for the ode $\dot{x} = Ax$ for when the control is zero. Identify all the equilibrium points and determine whether they are stable in any of the senses we've considered this semester.

Solution:



The equilibrium points are those with $Ax = 0$ or equivalently $x_2 = 0$. So every point on the x_1 axis is an equilibrium point. None of the equilibrium points are stable because a small ball around any one of them contains points with $x_2 \neq 0$ and from those points $|x(t)| \to \infty$.

(b) What is the controllable subspace, Σ_c?

Solution: Σ_c is the column span of the controllability matrix $\mathcal{C} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. So $\Sigma_c = \text{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which is also the x_1 axis.

(c) What is the unobservable subspace, $\Sigma_{\bar{o}}$?

Solution: $\Sigma_{\bar{o}}$ is the null space of the observability matrix $\mathcal{O} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. So $\Sigma_{\bar{o}} = \operatorname{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which is also the x_1 axis.

4. **[Problem 4 (24 points)]**
Suppose the dual of an LTI system with matrices A, B, C, D is equal to itself. That is, $A = -A^*, B = C^*, C = B^*, D = D^*$. (Hint: Think of all properties of dual systems.)

(a) If the system is controllable must it also be observable? Explain your answer.

Solution: Yes, because if the system is controllable its dual is observable which means the system is observable.

(b) Must the system be stable in any of the senses discussed in class? Explain your answer.

Solution: Recall that if x is a solution for A and z is a solution for the dual, then $x^*(t)z(t)$ is constant. Because the system is self dual we can take $x = z$ and conclude that $x^*(t)x(t) = \|x(t)\|^2$ is constant with respect to t. This implies that the system is Lyapunov stable (and not asymptotically stable).

Equivalently, $V(x) = \|x\|^2 = x^*x$ we find $\frac{d}{dt}V(x_t) = (Ax(t))^*x(t) + x(t)Ax(t) = x^*(t)(A^* + A)x(t) = 0$ so that V is a Lyapunov function showing that ϑ is Lyapunov stable by not asymptotically stable.

Note: As the system in problem 3 above shows, having purely imaginary eigenvalues (in that case both are zero) does not imply Lyapunov stability.

(c) What can be said about the eigenvalues of A? Explain your answer.

Solution: The eigenvalues of A are purely imaginary. If λ is an eigenvalue then there is a corresponding eigenvector v and $e^{At}v = e^{\lambda t}v$. By part (b) it follows that $|e^{\lambda t}| = 1$ for all t, or equivalently, λ is purely imaginary.

Another way to see this is to note that jA is Hermitian symmetric and Hermitian symmetric matrices have real eigenvalues.

Yet another way is to note that $Av = \lambda v$ implies that $v^*Av = \lambda\|v\|^2$. Also $-A^*v = \lambda v$ so that $v^*A = -\overline{\lambda}v^*$ and thus $v^*Av = -\overline{\lambda}\|v\|^2$. So $\lambda = -\overline{\lambda}$.