**Name:** ____________________ **NetID:** __________

## ECE 515 / ME 540 Fall 2024 Midterm 1, October 3, 2024

**75 min; 1 page notes 2-sided OK; closed book; no caculators; scan & upload to gradescope when done.**

1. **[Problem 1 (24 points)]**
   Consider the SISO nonlinear system with state equation $\dot{y} = \sin(2y + u) + e^{-3y+2u} - 1$.

   (a) Find the linear system obtained by linearizing about the equilibrium point $y_e = u_e = 0$.
   **Solution:** Since $\frac{d\sin(x)}{dx} = \cos(x)$ and $\cos(0) = 1$ and $\frac{de^x}{dx} = e^x$ and $e^0 = 1$ we get by the chain rule of calculus (ignoring higher order terms):
   $$\delta\dot{y} = 2\delta y + \delta u - 3\delta y + 2\delta u = -\delta y + 3\delta u$$

   (b) If the input $u$ is identically zero, in what sense(s), if any, is the equilibrium point 0 stable for the linearized system? Briefly explain your answer.
   **Solution:** It is asymptotically stable. For example, it is a linear system with negative eigenvalue -1. (For linear systems asymptotic stability and global asymptotic stability are the same thing so it is also globally asymptotically stable but no points will be deducted if that is not mentioned.)

   (c) If the input $u$ is identically zero, in what sense(s), if any, is the equilbrium point 0 stable for the original nonlinear system? Briefly explain your answer.
   **Solution:** It is also asymptotically stable for the original nonlinear system because the eigenvalue is negative. (See Theorem 4.7 in the notes.) There are other equilibrium points near positive values of $y$ such that $\sin(2y) = 1$ so the system is not globally asymptotically stable.

2. **[Problem 2 (20 points)]**
   Consider the dynamical system $\dot{x} = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix} x$.

   (a) What is the stable subspace $\Sigma_s$ of the system?
   **Solution:** Since $\det(Is - A) = (s+3)^2(s-1)$ the eigenvalues are -3, -3, 1. Moreover, the matrix is in Jordan canonical form with a $2 \times 2$ block with eigenvalue -3 and a $1 \times 1$ block with eigenvalue 2. The stable subspace is the span of eigenvectors and generalized eigenvectors corresponding to eigenvalues with negative real parts. The eigenvector for eigenvalue -3 is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and a choice of generalized eigenvector for eigenvalue -3 is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.
   These span the stable subspace, which is given by the set of vectors $x \in \mathbb{R}^3$ such that $x_3 = 0$.

   (b) Consider the new system given by the change of coordinates $\overline{x}(t) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} x(t)$.

   What is the stable subspace of the new system?

**Solution:** It is the stable space from part (a) transformed by the same change of coordinates. So it is the span of the first two columns of the coordinate change matrix, or, equivalently, the set of vectors $\bar{x}$ such that $\bar{x}_2 = \bar{x}_3$.

3. **[Problem 3 (16 points)]**
Suppose $A, B$ and $C$ are each $2\times 2$ matrices such that $A$ has eigenvalues 1,2, $B$ has eigenvalues $3, 4$, and $C$ has eigenvalues $5, 6$.

   (a) What are the eigenvalues of $e^{At}$ for $t \geq 0$?
   **Solution:** $e^{At}$ has eigenvalues $e^{t}$ and $e^{2t}$. (Think of Jordan canonical form to see that the eigenvalues of $e^{At}$ are given by $e^{\lambda_i t}$, where $\lambda_1, \ldots, \lambda_n$ are the possibly nondistinct eigenvalues of $A$.

   (b) What are the eigenvalues of the $4 \times 4$ matrix $M = \begin{pmatrix} A & 0_{2\times 2} \\ B & C \end{pmatrix}$?
   **Solution:** The union of the eigenvalues of $A$ and $C$: 1,2,4,6. (Thinking about the formula for det as a sum over permutations of products, we see that none of the product terms can contain any entries of the matrix $B$. So $\det(I\lambda - M) = \det(I\lambda - A)\det(I\lambda - C)$.)

4. **[Problem 4 (24 points)]**
Let $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \end{pmatrix}$.

   (a) What is the dimension of the column span of $A$?
   **Solution:** Two. The first two columns are linearly independent. The third column is the sum of the first two columns and the fourth column is the sum of the twice the first column and the second column.

   (b) What is the dimension of the row span of $A$
   **Solution:** The answer has to be the same as the answer to part (a): Two. Some calculation shows that the middle row is the average of the first and last rows.

   (c) What is the dimension of the null space of $A$?
   **Solution:** Two. The null space is the orthogonal complement to the vectors spanned by the rows transposed. So it has dimension equal to 4 - dimension(row span) which is two.

5. **[Problem 5 (16 points)]**
Suppose $v^1, v^2, v^3$ is a basis for a vector space $\mathcal{V}$ over the field of complex numbers with an inner product. Let $G$ denote the Gramian matrix with $i, j^{th}$ element given by $g_{i,j} = \langle v^i, v^j \rangle$.

   (a) Must $G$ be nonsingular? If yes, explain why. If no, give a counter example.
   **Solution:** Yes. Proof 1: Suppose $\beta$ is a $3 \times 1$ vector such that $G\beta = 0$. Then if we let $y = \beta_1 v^1 + \beta_2 v^2 + \beta_3 v^3$ we find that $\langle v^i, y\rangle = (G\beta)_i = 0$ for $1 \leq i \leq 3$. Thus, $y$ is orthogonal to all three basis vectors and hence is orthogonal to all vectors in $\mathcal{V}$, including itself. So it must be that $y = \vartheta$. Hence, since $v^1, v^2, v^3$ is a basis it must be that $\beta = \vartheta$. So the columns of $G$ are linearly independent; $G$ has full rank and is nonsingular.
   Proof 2: Let $V$ denote the matrix formed from the basis vectors: $V = \left[v^1 v^2 v^3\right]$. Since the columns of $V$ are linearly independent, $V$ has full rank 3 and is thus nonsingular. Note that $G = V^*V$, so that $G$ is the product of the two nonsingular matrices $V^*$ and $V$, and hence $G$ is nonsingular.

(b) Suppose $x$ is a vector in $\mathcal{V}$ and suppose that $\alpha = \begin{pmatrix} \langle v^1, x \rangle \\ \langle v^2, x \rangle \\ \langle v^3, x \rangle \end{pmatrix}$. Let $\beta$ denote the coordinate vector of $x$ with respect to the basis $v^1, v^2, v^3$. Derive an expression for $\beta$ in terms of $\alpha$ and $G$.

**Solution:** By the definition of $\beta$, $x = \beta_1 v^1 + \beta_2 v^2 + \beta_2 v^3$. Therefore, $\alpha_1 = \langle v^1, x \rangle = \langle v^1, \beta_1 v^1 + \beta_2 v^2 + \beta_2 v^3 \rangle = \beta_1 g_{1,1} + \beta_2 g_{1,2} + \beta_3 g_{1,3}$. More generally, $\alpha = G\beta$. Thus, $\beta = G^{-1}\alpha$.