Note: Problems (or parts of problems) marked with a star (★) are required for graduate students to receive 4 credit hours; undergraduate students who solve these problems will receive extra credit points.

Submission: Write your name, netid, and U for undergrad/G for grad in the upper right-hand corner of the first page of your written solutions. Typewritten solutions will receive 5 extra credit points.

Problems to be handed in

1 Let $X = (X_t)_{t\in\mathbb{R}}$ be a weakly stationary stochastic signal with zero mean and autocorrelation function $R_X(\tau) = \sigma^2 e^{-|\tau|}$. Let Y be the stochastic signal obtained from X via

$$Y_t = \frac{1}{2t_0}\int_{t-t_0}^{t+t_0} X_\tau \mathrm{d}\tau,$$

where $t_0 > 0$ is a fixed window size.

(a) Prove that Y is also weakly stationary.

(b) Find the mean m_Y and the power spectral density S_Y.

2 Let $X = (X_t)_{t\in\mathbb{R}}$ be a weakly stationary stochastic signal whose power spectral density $S_X(\omega)$ vanishes if $|\omega| \geq 2\pi B$. By analogy with the deterministic case, we say that X is *bandlimited* with bandwidth B Hz. Given such a signal, suppose we sample it at the Nyquist rate and generate the samples X_{kT}, for $k \in \mathbb{Z}$, where $T = \frac{1}{2B}$ is the Nyquist sampling period. Then the value of X_t at any t can be reconstructed via the sampling expansion

$$X_t = \sum_{k=-\infty}^{\infty} X_{kT}\operatorname{sinc}\Big(2B(t-kT)\Big),$$

where $\operatorname{sinc}(u) \triangleq \frac{\sin(\pi u)}{\pi u}$ is the sinc function. Now consider the situation where the samples X_{kT} are corrupted by noise before reconstruction. In this case, we will not be able to reconstruct X_t exactly. In this problem, we will analyze this situation and compute the expected squared error.

We will model the noise by a weakly stationary discrete-time stochastic signal $N = (N_k)_{k\in\mathbb{Z}}$ with zero mean $m_N(k) = \mathbf{E}[N_k] = 0$ and a given autocorrelation function $R_N(m) = \mathbf{E}[N_k N_{k+m}]$. Note that we are *not* assuming that N is independent of X. Each sample X_{kT} is corrupted to the noisy version $\tilde{X}_k \triangleq X_{kT} + N_k$, and then we attempt to reconstruct X_t from the noisy samples by

$$\begin{aligned}\widehat{X}_t &= \sum_{k=-\infty}^{\infty} \tilde{X}_k \operatorname{sinc}\Big(2B(t-kT)\Big) \\ &= \sum_{k=-\infty}^{\infty} (X_{kT} + N_k)\operatorname{sinc}\Big(2B(t-kT)\Big).\end{aligned}$$

We will examine the reconstruction error $\Delta_t \triangleq \mathbf{E}[(X_t - \widehat{X}_t)^2]$.

(a) Define the following continuous-time stochastic signal $W = (W_t)_{t\in\mathbb{R}}$:

$$W_t \triangleq \sum_{k=-\infty}^{\infty} N_k \,\mathrm{sinc}\Big(2B(t-kT)\Big),$$

and show that W is weakly stationary with $m_W = 0$ and

$$R_W(\tau) = \sum_{m=-\infty}^{\infty} R_N(m)\mathrm{sinc}\Big(2B(\tau+mT)\Big).$$

Hint: The formula

$$\mathrm{sinc}\Big(2B(t+\theta)\Big) = \sum_{k=-\infty}^{\infty} \mathrm{sinc}\Big(2B(t-kT)\Big)\mathrm{sinc}\Big(2B(kT+\theta)\Big)$$

may come in handy.

(b) Prove that $\Delta_t = \mathbf{E}[W_t^2]$ and use the result from part (a) to show that

$$\Delta_t = R_N(0).$$

3 In this problem, we will explore some properties of jointly Gaussian random variables.

(a) Recall that the characteristic function of a scalar random variable X is given by

$$\Phi_X(u) = \mathbf{E}[e^{\mathrm{i}uX}], \qquad u \in \mathbb{R}.$$

and that the joint characteristic function of a random vector $X = (X_t)_{t\in\{1,\ldots,n\}}$ is given by

$$\Phi_{X_1,\ldots,X_n}(u_1,\ldots,u_n) = \mathbf{E}[e^{\mathrm{i}(u_1X_1+\ldots+u_nX_n)}], \qquad u_1,\ldots,u_n \in \mathbb{R}.$$

Assume that $X_1,\ldots,X_n$ have a joint pdf $f_{X_1,\ldots,X_n}$. We say that $X_1,\ldots,X_n$ are independent random variables if

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = f_{X_1}(x_1)f_{X_2}(x_2)\ldots f_{X_n}(x_n),$$

where f_{X_i} denotes the marginal pdf of X_i. Prove that $X_1,\ldots,X_n$ are independent if and only if

$$\Phi_{X_1,\ldots,X_n}(u_1,\ldots,u_n) = \Phi_{X_1}(u_1)\Phi_{X_2}(u_2)\ldots\Phi_{X_n}(u_n)$$

for all $u_1,\ldots,u_n \in \mathbb{R}$.

(b) We say that $X_1,\ldots,X_n$ are uncorrelated if the covariance matrix C_X is diagonal, i.e., if $C_X(s,t) = \mathbf{E}[X_sX_t] - \mathbf{E}[X_s]\mathbf{E}[X_t] = 0$ for $s \neq t$. In general, uncorrelated random variables can still be dependent. Use the result from part (a) to prove that if $X_1,\ldots,X_n$ are uncorrelated and jointly Gaussian, then they are independent.

(c) Let X be a Gaussian random vector. In class, we have proved that the projection $a^T X$ of X onto any deterministic vector $a = (a_1, \dots, a_n)^T \in \mathbb{R}^n$ is Gaussian. Now consider an $m \times n$ matrix $A = (A_{ij})_{i,j \in \{1,\dots,n\}}$ and form the random vector $Y = AX$. Prove that Y is also a Gaussian random vector.

(d) Let X be a Gaussian random variable with mean 0 and variance σ^2. Let U be a Rademacher random variable (i.e., $\mathbf{P}[U = \pm 1] = \frac{1}{2}$) independent of X. Prove that $Y = UX$ is also Gaussian with mean 0 and variance σ^2, but X and Y are *not* jointly Gaussian.

Hint: Consider the sum $X + Y$.

4 Let X be a zero-mean stationary Gaussian stochastic signal. Compute the crosscorrelation function $R_{XY}(\tau)$ between X and Y, where $Y_t = g(X_t)$ with the following choices for g:

(a) The full-wave rectifier $g(x) = |x|$.

(b) The power-law detector $g(x) = x^p$ for $p \in \mathbb{N}$.

(c) The gating function $g(x) = u(x+1) - u(x-1)$.

5 (★) Poisson processes are used to model situations where discrete events happen at random times. For example, a Poisson process with rate λ can be used to model the number of customers arriving at a ticket counter in the airport, where λ is the average number of new customer arrivals per unit time.

(a) We will first consider the situation when there are several independent queues of customers arriving at the counter. Formally, let m be the number of queues, and for each $k \in \{1, \dots, m\}$ let $N^{(k)} = (N_t^{(k)})_{t \geq 0}$ be a Poisson process with rate λ_k. Thus, λ_k is the average number of customers arrivals per unit time via the kth queue. We assume that these Poisson processes are mutually independent. The total number of arrivals at the counter at time t is then

$$N_t = \sum_{k=1}^{m} N_t^{(k)}.$$

Prove that N is also a Poisson process and compute its arrival rate.

(b) Now we consider the opposite situation: customers arrive at a rate of λ, but each new customer independently decides to join one of two queues with respective probabilities p and $1-p$. That is, we have a Poisson process $N = (N_t)_{t \geq 0}$ with rate λ, and then we form two counting processes $N^{(1)} = (N_t)_{t \geq 0}$ and $N^{(2)} = (N_t^{(2)})_{t \geq 0}$, where $N^{(i)}$ counts the arrivals of customers for the ith queue.

Prove that $N^{(1)}$ and $N^{(2)}$ are also Poisson processes and compute their arrival rates.