Note: Problems (or parts of problems) marked with a star (★) are required for graduate students to receive 4 credit hours; undergraduate students who solve these problems will receive extra credit points.

Submission: Write your name, netid, and U for undergrad/G for grad in the upper right-hand corner of the first page of your written solutions. Typewritten solutions will receive 5 extra credit points.

Problems to be handed in

1 Recall the simple random walk on the integers: a particle starts at $X_0 = 0$ and, at each time $t = 1, 2, \ldots$ takes a unit step to the left or to the right with equal probability. In class, have proved the following fact: For any $t \in \mathbb{N}$ and any $x \in \mathbb{Z}$,

$$\mathbf{P}[X_t = x] = \frac{1}{2^t} N_t(x) := \begin{cases} \frac{1}{2^t}\binom{t}{\frac{x+t}{2}}, & \text{if } x = -t, 2-t, 4-t, \ldots, t-4, t-2, t \\ 0, & \text{otherwise.} \end{cases} \tag{0.1}$$

In other words, $N_t(x)$ is the number of ways in which the particle starting at the origin at time 0 can reach the point x at time t. Now, for $x \in \mathbb{N}$, let $N_t^+(x)$ denote the number of ways in which the particle starting at the origin at time 0 can reach the point x at time t without ever visiting any point $y \le 0$.

(a) Consider any two times $0 < s < t$ and any two points $x, y \in \mathbb{Z}$. Prove that the conditional probability that the particle will reach the point y at time t given that it was at the point x at time s is

$$\mathbf{P}[X_t = y | X_s = x] = \frac{1}{2^{t-s}} N_{t-s}(y - x).$$

(b) For $0 < s < t$ and for $x, y \in \mathbb{N}$, let $N^0_{s,t}(x, y)$ denote the number of ways the particle starting from $X_s = x$ will reach $X_t = y$ *and* visit the origin at some intermediate time $s \le r \le t$. Prove that

$$N^0_{s,t}(x, y) = N_{t-s}(x + y).$$

Hint: Figure 1 may be helpful.

(c) Use the results from parts (a) and (b) to prove that, for any $x \in \mathbb{N}$,

$$\mathbf{P}[X_1 > 0, X_2 > 0, \ldots, X_{t-1} > 0, X_t = x] = \frac{x}{t}\mathbf{P}[X_t = x].$$

Hint: The above probability is equal to $\frac{1}{2^t} N_t^+(x)$ (why?). Thus, you need to compute $N_t^+(x)$. To that end, observe that, since $X_0 = 0$, the only way for $X_1 > 0$ to happen would be to have $X_1 = 1$. Thus, you need to count the number of ways that the particle starting at $X_1 = 1$ can reach $X_t = x$ without ever visiting any $y \le 0$ in between.

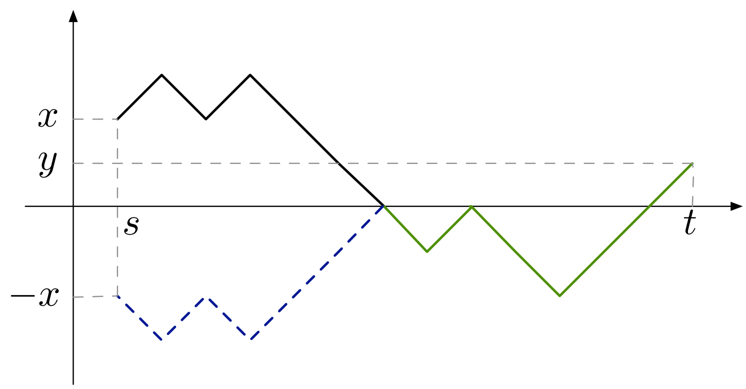

Figure 1: Illustration for Problem 1(b).

2 Recall the example of the sinusoidal signal with random amplitude and phase:

$$X_t = A\cos(2\pi f t + \Theta),$$

where the amplitude $A \sim \text{Uniform}(0,1)$ and the phase $\Theta \sim \text{Uniform}(0,2\pi)$ are independent random variables, and the frequency f (in Hz) is deterministic.

(a) Compute the mean function $m_X(t)$ and the variance function $\sigma_X^2(t)$.

(b) Compute the covariance function $C_X(s,t)$.

(c) What does the value of $C_X(t, t+1/f)$ tell you about this stochastic signal?

3 In class, we have derived the marginal distribution of the Wiener process at each time t using the DeMoivre–Laplace theorem. In this problem, we will develop an alternative approach via the *diffusion equation*.

(a) Fix arbitrary $\tau, h > 0$ and show that, for any $n \in \mathbb{Z}_+$ and any $m \in \mathbb{Z}$,

$$\mathbf{P}[X_{(n+1)\tau} = mh] - \mathbf{P}[X_{n\tau} = mh] = \frac{\mathbf{P}[X_{n\tau} = m(h-1)] - 2\mathbf{P}[X_{n\tau} = mh] + \mathbf{P}[X_{n\tau} = m(h+1)]}{2}.$$

(b) Let $f_t(x)$ denote the pdf of W_t. Let $t = n\tau$ and $x = mh$, where h and τ satisfy the constraint $h^2/\tau = D$. Using this and the relation derived in part (a), take the limit $\tau \to 0$ and show that f_t satisfies the following partial differential equation:

$$\frac{\partial}{\partial t} f_t(x) = \frac{D}{2}\frac{\partial^2}{\partial x^2} f_t(x).$$

(c) Verify by substitution that the solution of the above PDE with the initial condition $f_0(x) = \delta(x)$ [where $\delta(\cdot)$ is the unit impulse] is given by the Gaussian pdf

$$f_t(x) = \frac{1}{\sqrt{2\pi D t}} \exp\left(-\frac{x^2}{2Dt}\right).$$

4 In this problem, we will explore some properties of the Wiener process. A *standard Wiener process* $W = (W_t)_{t\geq 0}$ is a Wiener process with $D = 1$.

(a) Prove that the covariance function of W is given by $C_X(s,t) = \min\{s,t\}$.

(b) Let $c > 0$ be a fixed positive constant, and define another stochastic signal $Y = (Y_t)_{t\geq 0}$ by letting $Y_t = \frac{1}{\sqrt{c}} W_{ct}$. Prove that Y is also a standard Wiener process. (This shows that the sample paths of a Wiener process look the same at every time scale — as long as we rescale space to compensate for the time scaling.)

(c) Again, let $c > 0$ be a fixed constsant, and define another stochastic signal $Z = (Z_t)_{t\geq 0}$ by letting $Z_t = W_{t+c} - W_c$. Prove that Z is a standard Wiener process, and that it is independent of $(W_t)_{0\leq t\leq c}$. (This shows that the Wiener process can be thought of continually restarting anew from its current position.)

5 ($\star$) We continue with the set-up from Problem 4. For $b > 0$, define the *hitting time*

$$\tau_b \triangleq \min\{t \geq 0 : W_t \geq b\},$$

i.e., the first time when the particle is at a distance b away from the origin (it may, and will, go below b later, and then above b, and then below, and so on). This is a random variable, since it depends on the random path of W_t. You will prove the following neat formula:

$$\mathbf{P}[\tau_b \leq t] = 2Q\left(\frac{b}{\sqrt{t}}\right), \qquad t \geq 0$$

where $Q(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-x^2/2} \mathrm{d}x$ is the complementary Gaussian cdf.

(a) By the law of total probability,

$$\mathbf{P}[\tau_b \leq t] = \mathbf{P}[\tau_b \leq t, W_t \leq b] + \mathbf{P}[\tau_b \leq t, W_t > b].$$

Now argue that the events $\{\tau_b \leq t, W_t > b\}$ and $\{W_t > b\}$ are equivalent (the continuity of W_t as a function of t is crucial for this to hold), and conclude from this that

$$\mathbf{P}[\tau_b \leq t] = \mathbf{P}[W_t \leq b | \tau_b \leq t]\, \mathbf{P}[\tau_b \leq t] + Q\left(\frac{b}{\sqrt{t}}\right).$$

(b) Again, using the continuity of W_t in t, argue that $\mathbf{P}[W_t \leq b | \tau_b \leq t] = \frac{1}{2}$ (it may be helpful to draw a picture).

(c) Put all the pieces together to obtain the formula we seek.