Note: Problems marked with a star (⋆) are required for graduate students to receive 4 credit hours; undergraduate students who solve these problems will receive extra credit points.

Submission: Write your name, netid, and U for undergrad/G for grad in the upper right-hand corner of the first page of your written solutions. Typewritten solutions will receive 5 extra credit points.

Problems to be handed in

1 (Probability review)

(i) Let $(X, Y) \in \mathbb{R}^2$ be a point chosen uniformly at random from the unit disk

$$\mathbb{D} := \left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \right\}.$$

Compute the marginal pdf f_X of X and the expected value $\mathbf{E}[\sqrt{X^2 + Y^2}]$.

(ii) Let X be a real-valued random variable with finite second moment: $\mathbf{E}[X^2] < \infty$. Consider the function $f(c) = \mathbf{E}[(X - c)^2]$. Find the value of c that minimizes f and compute the corresponding minimum value of f.

(iii) Two real-valued random variables U and V are called uncorrelated if

$$\mathrm{Cov}(U, V) := \mathbf{E}[(U - \mathbf{E}[U])(V - \mathbf{E}[V])] = 0.$$

Let $U = \sin X$ and $V = \cos X$, where $X \sim \mathrm{Unif}(-\pi, \pi)$. Are U and V uncorrelated? Are they independent? Show all your work and justify all your answers!

2 In class, we saw how to generate a Bernoulli(p) random variable from a random variable U uniformly distributed on the unit interval $[0, 1]$: if $0 \leq U < p$, output 1; else, if $p \leq U < 1$, output 0.

(i) Modify the above procedure for the case when, instead of $U \sim \mathrm{Uniform}(0, 1)$, we can generate a random variable V with cdf given by

$$F_V(a) = \begin{cases} 0, & a < 0 \\ a^2, & 0 \leq a < 1 \\ 1, & a \geq 1 \end{cases}.$$

Hint: Can you find a function $f : \mathbb{R} \to \mathbb{R}$, such that $f(V) \sim \mathrm{Uniform}(0, 1)$?

(ii) Now suppose that we wish to generate a random variable X taking values in the finite set $\{1, \ldots, n\}$ with probabilities $p_i := \mathbf{P}[X = i]$. Construct a procedure that takes $U \sim \mathrm{Uniform}(0, 1)$ as the input and generates a sample of X as the output.

(iii) Finally, modify your procedure from part (ii) to work with input V from part (i).

3 Consider a Markov chain $X = (X_t)_{t\in\mathbb{Z}_+}$ with the ternary state space $\mathsf{X} = \{1,2,3\}$ and with the transition probability matrix

$$M = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix}.$$

This constitutes a declarative description of X. Construct an imperative description of X in the form $X_{t+1} = f(X_t, U_t)$, where the U_t's are i.i.d. Uniform$(0,1)$ random variables.

4 Prove that any discrete-time Markov chain has the following property:

$$\begin{aligned}\mathbf{P}[X_{t_{n+1}} = y_1, X_{t_{n+2}} = y_2, \ldots, X_{t_{n+m}} = y_m | X_{t_1} = x_1, X_{t_2} = x_2, \ldots, X_{t_n} = x_n] \\ = \mathbf{P}[X_{t_{n+1}} = y_1, X_{t_{n+2}} = y_2, \ldots, X_{t_{n+m}} = y_m | X_{t_n} = x_n]\end{aligned}$$

for all $t_1 < t_2 < \ldots < t_n < t_{n+1} < \ldots < t_{n+m}$ and for all states $x_0, \ldots, x_t, y_1, \ldots, y_m$. This is why we say that, in a Markov chain, the future is conditionally independent of the past given the present.

Hint: Pass to a suitable imperative description of the form $X_{t+1} = f(X_t, U_t)$ and iterate.

5 ($\star$) The *moment-generating function* (MGF) of a real-valued random variable X is defined as $M(\lambda) := \mathbf{E}[e^{\lambda X}]$.

(i) Find the MGF of a *Rademacher random variable*, i.e., $\mathbf{P}[X = -1] = \mathbf{P}[X = +1] = \frac{1}{2}$.

(ii) Find the MGF of a Gaussian random variable with mean μ and variance σ^2.

(iii) Use the result of part (ii) to prove the following: If $X \sim N(\mu, \sigma^2)$, then, for any $t > 0$,

$$\mathbf{P}[X \geq \mu + t] \leq e^{-t^2/2\sigma^2}.$$

Hint: Note that, for any $\lambda > 0$, $\mathbf{P}[X \geq \mu + t] = \mathbf{P}[e^{\lambda X} \geq e^{\lambda(\mu+t)}]$.

(iv) Prove the following inequality for any two real-valued random variables X and Y:

$$|\mathrm{Cov}(X,Y)|^2 \leq \mathrm{Var}[X] \cdot \mathrm{Var}[Y].$$

Hint: Assume $\mathbf{E}[X] = \mathbf{E}[Y] = 0$ and observe that $\mathbf{E}[(X + \alpha Y)^2] \geq 0$ for any $\alpha \in \mathbb{R}$.