

Question 1. Convex functions. Prove or disprove convexity on the stated domain.

(a) $f(t) = t^2, t \in \mathbb{R}$.

(b) $h(x) = x_1x_2, x \in \mathbb{R}^2$.

(c) $q(x) = (x_1 - 2x_2)^2 + x_2^2, x \in \mathbb{R}^2$.

Question 2. Convex sets. Prove or disprove convexity using a lecture criterion.

(a) $D = \{x \in \mathbb{R}^2 : x_1, x_2 \geq 0, \|x\|_2 \leq 1\}.$

(b) $T = \{(s, x) \in \mathbb{R} \times \mathbb{R}^n : \|x\|_2 \leq s\}.$

Question 3. Matrix rank restriction. For $n \geq 2$, prove or disprove convexity of

$$K = \{X \in \mathbb{S}^n : X \succeq 0, \text{rank}(X) \leq 1\}.$$

Possible student answer (wrong):

"The set $\{X : X \succeq 0\}$ is convex, but the function $\text{rank}(X)$ is nonconvex, so its level set is nonconvex. Therefore the union is nonconvex."

Question 4. Optimality conditions. Let

$$q(x) = \frac{1}{4}\|x\|^4 - \frac{1}{2}\|x\|^2, \quad x \in \mathbb{R}^2.$$

Compute the gradient and Hessian. Find all first- and second-order critical points, prove that q is not convex, and determine all global minimizers.

Question 5. Descent and first-order accuracy. Let $B = B^T \succ 0$, $\|u\|_B^2 = u^T B u$, and $\|u\|_{B^{-1}}^2 = u^T B^{-1} u$. Suppose f is continuously differentiable and globally L -smooth in these norms: $\|\nabla f(x) - \nabla f(y)\|_B \leq L\|x - y\|_{B^{-1}}$, with $L > 0$. You may use the quadratic upper bound

$$f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L}{2} \|y - x\|_{B^{-1}}^2.$$

For $x_{k+1} = x_k - \alpha B \nabla f(x_k)$, derive the descent lemma and optimize its guaranteed decrease. Assuming $f^* = \inf_x f(x) > -\infty$, prove for $N \geq 1$

$$\min_{0 \leq k < N} \|\nabla f(x_k)\|_B^2 \leq \frac{2L[f(x_0) - f^*]}{N}.$$

Question 6. Strong convexity, PL, and linear convergence. Under the same smoothness assumption, suppose $f^\star = f(x^\star) = \min_x f(x)$ and $0 < \mu \leq L$. Assume for every x, y that

$$f(y) \geq f(x) + \nabla f(x)^T(y - x) + \frac{\mu}{2}\|y - x\|_{B^{-1}}^2.$$

Prove the PL inequality $\frac{1}{2}\|\nabla f(x)\|_B^2 \geq \mu[f(x) - f^\star]$, then derive the objective-gap convergence rate for step $\alpha = 1/L$.

Question 7. Newton's method: models and local convergence.

(a) Minimize the quadratic Taylor model at x to derive the full Newton step. Show that one full step minimizes $q(x) = c + b^T x + \frac{1}{2}x^T A x$, $A \succ 0$, from any starting point.

(b) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be twice continuously differentiable, with

$$\mu I \preceq \nabla^2 f(x) \preceq L_1 I, \quad \|\nabla^2 f(u) - \nabla^2 f(w)\|_{\text{op}} \leq L_2 \|u - w\|,$$

where $\mu, L_2 > 0$. Using the fundamental theorem of calculus along the segment from x^* to x , prove

$$\|x + v - x^*\| \leq \frac{L_2}{2\mu} \|x - x^*\|^2, \quad v = -\nabla^2 f(x)^{-1} \nabla f(x).$$

Explain why this is a local convergence guarantee.

