

Learning objectives. By the end of class, you should be able to

1. state and prove the weak alternative, and construct infeasibility certificates using nonnegative Lagrangian multipliers;
2. derive the dual function $g(y)$ and use it to certify infeasibility;
3. explain why the weak alternative need not be strong, and why the dual function may fail to detect an existing certificate;
4. state and apply Farkas' lemma and the Fredholm alternative to certify infeasibility of linear systems.

1 Motivation

This section reviews constrained optimization and explains why we need proofs that constraints cannot be met.

Most practical problems come with constraints.

- **Power dispatch.** Outputs $p_i \leq c_i$ must meet the demand, $p_1 + \dots + p_k \geq d$, at minimum fuel cost.
- **Portfolio.** Fractions $x_i \geq 0$ with $\sum_i x_i = 1$ must reach a return $\mu^T x \geq r$, at minimum risk $x^T \Sigma x$.
- **Motion planning.** A drone avoids an obstacle, $\|x_t - c\| \geq r$, with bounded steps, $\|x_{t+1} - x_t\| \leq v$.
- **Sparse regression.** Least squares with a budget $\|\theta\|_1 \leq \tau$ that sets many coefficients to zero.

Constraints can conflict. If the demand exceeds the total capacity, or no investment returns r , then no solution exists. A solver must then report that the problem is infeasible, and we would like a proof.

1.1 Feasibility and optimal value

In general, the **constrained problem** is

$$\begin{array}{ll} \min_x & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0 \quad (i = 1, \dots, m). \end{array}$$

Its **feasible set** is

$$\Omega := \{x \in \mathbb{R}^n : f_i(x) \leq 0, \ i = 1, \dots, m\},$$

and its **optimal value** is

$$p^* := \inf_{x \in \Omega} f_0(x).$$

An equality $f_i(x) = 0$ is the pair $f_i(x) \leq 0$, $-f_i(x) \leq 0$, as in Lecture 1, so the problem needs only inequalities.

Question 1. What does it mean for a point x to be feasible? Infeasible?

Question 2. What does it mean for the problem to be feasible? Infeasible? What is p^* in each case?

1.2 How to solve constrained problems

We solve them by generalizing the tools of Blocks 1–3.

Question 3. What is the constrained counterpart of each tool?

Blocks 1–3

Blocks 4–6

Every x is feasible.

First-order optimality

$$\nabla f(x) = 0.$$

Second-order optimality

$$\nabla f(x) = 0, \nabla^2 f(x) \succeq 0.$$

Certify global optimality
by convexity.

Gradient descent, Newton.

The first four rows each ask whether a system has a solution. Is there a point in Ω ? Is there an improving direction? Is there a point better than the one we have? In Blocks 1–3, x is first-order optimal when no direction v has $\nabla f(x)^T v < 0$, and $\nabla f(x) = 0$ proves that no such v exists. A solution of a system is checked by substitution. A claim that no solution exists is a statement about every candidate, and it needs a proof. This lecture is about such proofs, called **certificates of infeasibility**.

2 The weak alternative

The following holds for any functions, convex or not, smooth or not.

Theorem 1. Weak alternative. Let $f_1, \dots, f_m : \mathbb{R}^n \rightarrow \mathbb{R}$ be any functions. At most one of the following holds.

- (i) There is a feasible point $x \in \mathbb{R}^n$ such that $f_i(x) \leq 0$ for all $i \in \{1, \dots, m\}$.
- (ii) There are multipliers $y \in \mathbb{R}_+^m$ with

$$\mathcal{L}(x, y) = \sum y_i f_i(x) + \dots + y_m f_m(x) > 0 \text{ for all } x \in \mathbb{R}^n.$$

The function $\mathcal{L}(x, y)$ is called the **Lagrangian**.

Multipliers $y \in \mathbb{R}_+^m$ satisfying $\mathcal{L}(x, y) > 0$ for all x are **infeasible certificate** for $\Omega = \emptyset$.

Question 4. Why not require $\mathcal{L}(x, y) \geq 0$?

Example 2. Power dispatch. Two generators have capacities 3 and 5 MW but must meet a demand of 10 MW. Obviously there is no dispatch with $p_1 \leq 3$, $p_2 \leq 5$, and $p_1 + p_2 \geq 10$. Construct an infeasibility certificate.

3 Finding certificates by duality

Original problem; call it the **primal**

$$\text{find } x \in \mathbb{R}^n \text{ s.t. } f_i(x) \leq 0 \quad \text{for all } i.$$

Certify infeasibility by solving yet another feasibility problem

$$\text{find } y \in \mathbb{R}_+^m \text{ s.t. } \mathcal{L}(x, y) = \sum_{i=1}^m y_i f_i(x) > 0 \text{ for all } x \in \mathbb{R}^n.$$

Systematic way to find y ? Construct the **dual function**:

$$g(y) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, y) > 0 \implies \mathcal{L}(x, y) > 0 \text{ for all } x \in \mathbb{R}^n.$$

Example 3. Certify that the following set is infeasible:

$$\Omega = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1, x_1 + x_2 \geq 2\}.$$

Systematic way to find $y \geq 0$ such that $g(y) > 0$?

Solve the **dual problem**

$$\sup_{y \in \mathbb{R}_+^m} g(y) > 0 \implies \text{exists } y \text{ s.t. } \mathcal{L}(x, y) > 0 \text{ for all } x \in \mathbb{R}^n.$$

Note: Interpret unbounded as $\sup_{y \in \mathbb{R}_+^m} g(y) = +\infty$.

The function g is always concave, even when $f_i(x)$ nonconvex. Dual problem is always convex optimization.

Example 4. Certify that the following set is infeasible

$$\Omega = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1, x_1 + x_2 \geq 2\}$$

by solving the dual problem.

4 Drawbacks of weak alternatives

We know *at most* one of the following is feasible

$$(P) \quad \text{find } x \in \mathbb{R}^n \text{ s.t. } f_i(x) \leq 0 \quad \text{for all } i$$

$$(D) \quad \text{find } y \in \mathbb{R}_+^m \text{ s.t. } g(y) = \inf_{x \in \mathbb{R}^n} \sum_{i=1}^m y_i f_i(x) > 0.$$

Trouble is *both can be infeasible*.

Question 5. Is $\Omega = \{x \in \mathbb{R} : x \leq 0, x^3 \geq 1\}$ feasible?

Question 6. What is the dual function?

Question 7. Does the dual problem certify infeasibility?

Question 8. Does Ω have an infeasibility certificate?

There is also a subtle difference between the dual problem and the weak alternative. Consider

$$\Omega = \{x \in \mathbb{R} : \exp(x) \leq 0\}.$$

Question 9. Is Ω feasible?

Question 10. What is the dual function?

Question 11. Does the dual problem certify infeasibility?

Question 12. Does there exist infeasibility certificate?

Conclusion: The dual is weaker than the weak alternative:

$$\sup_{y \in \mathbb{R}_+^m} g(y) > 0 \iff \text{exists } y \in \mathbb{R}_+^m \text{ s.t. } g(y) > 0$$

$$\implies \text{exists } y \in \mathbb{R}_+^m \text{ s.t. } L(x, y) > 0 \text{ for all } x \in \mathbb{R}^n$$

but the final $\implies$ does not go the other way.

5 The strong alternative

Weak alternative says *at most* one of the following is feasible

$$(P) \quad \text{find } x \in \mathbb{R}^n \text{ s.t. } f_i(x) \leq 0 \quad \text{for all } i$$

$$(D) \quad \text{find } y \in \mathbb{R}_+^m \text{ s.t. } g(y) = \inf_{x \in \mathbb{R}^n} \sum_{i=1}^m y_i f_i(x) > 0.$$

Under **constraint qualification (CQ)**, which are basically additional assumptions on $f_1, \dots, f_m$, strong alternative holds, and **exactly one** is feasible.

Theorem 5. Linear CQ. Let every f_i be affine, $f_i(x) = a_i^T x - b_i$. Then exactly one of (P) and (D) is feasible.

This simple-looking result is one of the most consequential results in optimization.

Question 13. Suppose $f_i(x) = a_i^T x - b_i$. When is the dual problem feasible?

We have derived the following famous results from duality.

Theorem 6. Farkas Lemma. Exactly one of the following is feasible

$$(P) \quad \text{find } x \in \mathbb{R}^n \text{ s.t. } Ax \leq b$$

$$(D) \quad \text{find } y \in \mathbb{R}_+^m \text{ s.t. } A^T y = 0, \quad b^T y < 0.$$

The vast majority of scheduling, assignment, logistic problems posed in this form and solved by feasibility algorithms.

Example 7. Power dispatch, again. Two generators have capacities 3 and 5 MW but must meet a demand of 10 MW. Obviously there is no dispatch with $p_1 \leq 3$, $p_2 \leq 5$, and $p_1 + p_2 \geq 10$. Construct an infeasibility certificate.

Theorem 8. Fredholm Alternative. Exactly one of the following is feasible

(P) find $x \in \mathbb{R}^n$ s.t. $Ax = b$

(D) find $z \in \mathbb{R}^m$ s.t. $A^T z = 0$, $b^T z \neq 0$.

6 Summary

Feasibility. A point is feasible if it satisfies every constraint. A problem is feasible if $\Omega \neq \emptyset$, which one point proves, and infeasible if $\Omega = \emptyset$, which requires ruling out every possible point.

The weak alternative. For arbitrary constraints $f_i(x) \leq 0$, at most one of the following exists: a feasible point x , or nonnegative multipliers $y \geq 0$ such that $\mathcal{L}(x, y) = \sum_{i=1}^m y_i f_i(x) > 0$ for all x . Such multipliers certify infeasibility because every feasible point would make $\mathcal{L}(x, y) \leq 0$.

Duality and its limitations. The dual function $g(y) = \inf_x \mathcal{L}(x, y)$ is always concave. Finding $y \geq 0$ with $g(y) > 0$ certifies infeasibility, but the converse can fail: a certificate may exist even when $g(y) = 0$. More fundamentally, an infeasible nonlinear system may admit no Lagrangian certificate at all.

The strong alternative. For linear inequalities, Farkas' lemma guarantees exactly one of $Ax \leq b$ or $y \geq 0$ with $A^T y = 0$ and $b^T y < 0$. The latter is an explicit certificate of infeasibility. Rewriting equalities as pairs of inequalities yields the Fredholm alternative: exactly one of $Ax = b$ or $A^T z = 0$ with $b^T z \neq 0$.