1 Constraint qualifications

Each item gives constraints

$$f_i(x) \le 0 \ \ \forall i \in \{1,\dots,m\}, \qquad h_j(x) = 0 \ \ \forall j \in \{1,\dots,p\},$$

and a feasible point x. Let $\mathcal{A}(x) = \{i : f_i(x) = 0\}$, and consider

$$\sum_{i\in\mathcal{A}(x)} y_i \nabla f_i(x) + \sum_{j=1}^{p} z_j \nabla h_j(x) = 0. \tag{$*$}$$

1. **LICQ** holds at x if the only solution of $(*)$ is $y = 0$, $z = 0$.
2. **MFCQ** holds at x if the only solution of $(*)$ with $y \ge 0$ is $y = 0$, $z = 0$.

For each item, decide whether LICQ and MFCQ hold at x, and whether Slater's condition holds.

Worked example 1.1

$x_1 \ge 0$, $x_2 \ge 0$, $x_1 + x_2 \ge 0$, at $x = (0,0)$.

Solution.

Step 1: active set. Write $f_1 = -x_1$, $f_2 = -x_2$, and $f_3 = -x_1 - x_2$. At $x = (0,0)$, $f_1 = f_2 = f_3 = 0$, so $\mathcal{A}(x) = \{1,2,3\}$. There are no equalities.

Step 2: solve $(*)$**.** The gradients are $\nabla f_1 = (-1,0)$, $\nabla f_2 = (0,-1)$, and $\nabla f_3 = (-1,-1)$. So $(*)$ reads

$$y_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + y_3 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = 0, \qquad \text{that is,} \qquad -y_1 - y_3 = 0, \quad -y_2 - y_3 = 0.$$

Let $y_3 = t$. The two equations give $y_1 = -t$ and $y_2 = -t$. So the solutions are $y = t(-1,-1,1)$ for $t \in \mathbb{R}$.

Step 3: LICQ. The choice $t = 1$ gives the nonzero solution $y = (-1,-1,1)$. So LICQ fails.

Step 4: MFCQ. A solution with $y \ge 0$ needs $y_1 = -t \ge 0$ and $y_3 = t \ge 0$. So $t = 0$, and $y = 0$ is the only solution with $y \ge 0$. So MFCQ holds.

Step 5: Slater's condition. All three constraints are affine, hence convex. At $(1,1)$, $f_1 = -1 < 0$, $f_2 = -1 < 0$, and $f_3 = -2 < 0$. So Slater's condition holds.

1. $x_1^2 + x_2^2 \le 2$, $x_1 = x_2$, at $x = (1,1)$.
2. $x_2 \ge x_1^2$, $x_2 \ge 2x_1^2$, at $x = (0,0)$.
3. $x_1^2 + x_2^2 \ge 1$, at $x = (1,0)$.
4. $x^2 \le 0$, with $x \in \mathbb{R}$, at $x = 0$.
5. $x_2 \le x_1^3$, $x_2 \ge 0$, at $x = (0,0)$.
6. $x_1 + x_2 \le 1$, $x_1 + x_2 \ge 1$, at $x = (1,0)$.

2 Convex problems through KKT

Consider

$$\min_x \ f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0 \ \ \forall i \in \{1,\dots,m\}, \qquad h_j(x) = 0 \ \ \forall j \in \{1,\dots,p\}.$$

A feasible x with multipliers $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^p$ satisfies the KKT conditions if

$$\nabla f_0(x) + \sum_{i=1}^{m} y_i \nabla f_i(x) + \sum_{j=1}^{p} z_j \nabla h_j(x) = 0, \qquad y \ge 0, \qquad y_i f_i(x) = 0 \ \ \forall i.$$

For each problem, complete the following steps.

1. Check Slater's condition.
2. Write the KKT conditions.
3. Solve the KKT conditions.

Worked example 2.1

$\min\ (x_1-3)^2 + x_2^2$ s.t. $x_1 \le 2$.

Solution.

Step 1: Slater's condition. Write $f_1 = x_1 - 2$. $\nabla^2 f_0 = 2I \succeq 0$, so f_0 is convex, and f_1 is affine. At $(0,0)$, $f_1 = -2 < 0$. So Slater's condition holds.

Step 2: KKT conditions. The gradients are $\nabla f_0 = (2(x_1-3), 2x_2)$ and $\nabla f_1 = (1,0)$. The KKT conditions are

$$\begin{aligned} \text{stationarity:} \quad & 2(x_1-3) + y_1 = 0, \quad 2x_2 = 0, \\ \text{sign:} \quad & y_1 \ge 0, \\ \text{complementary slackness:} \quad & y_1(x_1-2) = 0, \\ \text{feasibility:} \quad & x_1 - 2 \le 0. \end{aligned}$$

Step 3: solve. Complementary slackness requires $y_1 = 0$ or $f_1(x) = 0$. This gives two cases. In each case, solve the equations, then check the sign and feasibility conditions.

Case 1: $y_1 = 0$. Stationarity gives $x_1 = 3$ and $x_2 = 0$. Then $f_1 = 1 > 0$. Infeasible.

Case 2: $f_1 = 0$. Then $x_1 = 2$. Stationarity gives $y_1 = -2(2-3) = 2 \ge 0$ and $x_2 = 0$. All conditions hold.

The only KKT point is $x = (2,0)$ with $y_1 = 2$. The problem is convex and Slater's condition holds, so x is a global minimizer, with $f_0(x) = 1$.

1. $\min\ x_1^2 + x_2^2$ s.t. $x_1 + x_2 \ge 1$.
2. $\min\ 3x_1 + 4x_2$ s.t. $x_1^2 + x_2^2 \le 25$.
3. $\min\ x_1^2 + x_2^2 + x_3^2$ s.t. $x_1 \ge 2$, $x_1 + x_2 + x_3 = 3$.
4. $\min\ -\log x_1 - \log x_2$ s.t. $x_1 + 2x_2 \le 4$, over the domain $x > 0$.
5. **(G)** $\min\ (x_1-2)^2 + (x_2-2)^2$ s.t. $x_1^2 + x_2^2 \le 2$.

3 Nonconvex problems through KKT

Consider

$$\min_x \ f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0 \ \ \forall i \in \{1,\dots,m\}, \qquad h_j(x) = 0 \ \ \forall j \in \{1,\dots,p\}.$$

A feasible x with multipliers $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^p$ satisfies the KKT conditions if

$$\nabla f_0(x) + \sum_{i=1}^{m} y_i \nabla f_i(x) + \sum_{j=1}^{p} z_j \nabla h_j(x) = 0, \qquad y \ge 0, \qquad y_i f_i(x) = 0 \ \ \forall i.$$

In each problem below, the feasible set is compact and LICQ holds at every feasible point. For each problem, complete the following steps.

1. Write the KKT conditions.
2. Find all KKT points.
3. Find the global minimizer.

Worked example 3.1

min $x_1 x_2$ s.t. $x_1^2 + x_2^2 \le 2$.

Solution.

Step 1: KKT conditions. Write $f_1 = x_1^2 + x_2^2 - 2$. The gradients are $\nabla f_0 = (x_2, x_1)$ and $\nabla f_1 = (2x_1, 2x_2)$. The KKT conditions are

$$\begin{aligned} \text{stationarity:} \quad & x_2 + 2y_1 x_1 = 0, \quad x_1 + 2y_1 x_2 = 0, \\ \text{sign:} \quad & y_1 \ge 0, \\ \text{complementary slackness:} \quad & y_1(x_1^2 + x_2^2 - 2) = 0, \\ \text{feasibility:} \quad & x_1^2 + x_2^2 - 2 \le 0. \end{aligned}$$

Step 2: KKT points. Complementary slackness requires $y_1 = 0$ or $f_1(x) = 0$. This gives two cases.

Case 1: $y_1 = 0$. Stationarity gives $x_2 = 0$ and $x_1 = 0$. Then $f_1 = -2 \le 0$. So $x = (0,0)$ with $y_1 = 0$ is a KKT point.

Case 2: $f_1 = 0$. The first stationarity equation gives $x_2 = -2y_1 x_1$. Substituting into the second gives $x_1 - 4y_1^2 x_1 = 0$, that is, $x_1(1 - 4y_1^2) = 0$. If $x_1 = 0$, then $x_2 = 0$, which contradicts $x_1^2 + x_2^2 = 2$. So $1 - 4y_1^2 = 0$ and $y_1 = \pm\frac{1}{2}$. For $y_1 = -\frac{1}{2}$, the sign condition fails. For $y_1 = \frac{1}{2}$, $x_2 = -x_1$, and $x_1^2 + x_2^2 = 2$ gives $2x_1^2 = 2$, so $x_1 = \pm 1$. So $x = (1,-1)$ and $x = (-1,1)$ with $y_1 = \frac{1}{2}$ are KKT points.

Step 3: global minimizer. The feasible set is compact, so a global minimizer exists. LICQ holds at every feasible point, so the global minimizer is one of the KKT points. The values are

$$f_0(0,0) = 0, \qquad f_0(1,-1) = -1, \qquad f_0(-1,1) = -1.$$

So the global minimizers are $(1,-1)$ and $(-1,1)$, with $p^\star = -1$.

1. min $x_1^2 - x_2^2$ s.t. $x_1^2 + x_2^2 \le 1$.
2. min $x_1 + 2x_2$ s.t. $x_1^2 + x_2^2 = 5$.
3. min $x_1^2 + 2x_2^2$ s.t. $x_1^2 + x_2^2 = 1$.
4. **(G)** min $x_1 + 2x_2 + 2x_3$ s.t. $x_1^2 + x_2^2 + x_3^2 = 9$.

4 Second-order conditions

Consider

$$\min_x \ f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0 \ \ \forall i \in \{1,\dots,m\}, \qquad h_j(x) = 0 \ \ \forall j \in \{1,\dots,p\},$$

with Lagrangian

$$\mathscr{L}(x,y,z) = f_0(x) + \sum_{i=1}^{m} y_i f_i(x) + \sum_{j=1}^{p} z_j h_j(x).$$

At a KKT point $(\hat{x}, y, z)$, the critical cone is

$$\mathcal{C}(\hat{x},y) = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} \nabla h_j(\hat{x})^T d = 0 & \forall j, \\ \nabla f_i(\hat{x})^T d = 0 & \forall i \text{ with } y_i > 0, \\ \nabla f_i(\hat{x})^T d \le 0 & \forall i \text{ with } f_i(\hat{x}) = 0 \text{ and } y_i = 0 \end{array} \right\}.$$

Each item gives a KKT point $\hat{x}$ with its multipliers, and LICQ holds at $\hat{x}$. For each item, complete the following steps.

1. Write the critical cone explicitly in terms of $d_1, \dots, d_n$.
2. Check the second-order necessary condition.
3. Check the second-order sufficient condition.

Worked example 4.1

min $x_1 + x_2$ s.t. $x_1^2 + x_2^2 \le 2$, $x_1 \le -1$, at $\hat{x} = (-1,-1)$ with $y = (\frac{1}{2}, 0)$.

Solution.

Step 1: critical cone. Write $f_1 = x_1^2 + x_2^2 - 2$ and $f_2 = x_1 + 1$. At $\hat{x}$, $f_1 = 0$ and $f_2 = 0$, so both are active. The gradients are $\nabla f_1(\hat{x}) = (2x_1, 2x_2) = (-2,-2)$ and $\nabla f_2 = (1,0)$. Since $y_1 = \frac{1}{2} > 0$, constraint 1 gives an equation. Since $y_2 = 0$, constraint 2 gives an inequality. So

$$\mathcal{C}(\hat{x},y) = \{d : -2d_1 - 2d_2 = 0, \ d_1 \le 0\}.$$

The equation gives $d_2 = -d_1$. Let $d_1 = -t$. The inequality $d_1 \le 0$ becomes $t \ge 0$. So $\mathcal{C}(\hat{x},y) = \{(-t,t) : t \ge 0\}$.

Step 2: necessary condition. The Hessians are $\nabla^2 f_0 = 0$, $\nabla^2 f_1 = 2I$, and $\nabla^2 f_2 = 0$. So

$$\nabla^2_{xx}\mathscr{L}(\hat{x},y,z) = \nabla^2 f_0 + y_1 \nabla^2 f_1 + y_2 \nabla^2 f_2 = 0 + \tfrac{1}{2}(2I) + 0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For $d = (-t, t)$,

$$d^T \nabla^2_{xx}\mathscr{L}\, d = d_1^2 + d_2^2 = (-t)^2 + t^2 = 2t^2 \ge 0.$$

So the necessary condition holds.

Step 3: sufficient condition. A nonzero $d \in \mathcal{C}(\hat{x},y)$ has $t > 0$, so $d^T \nabla^2_{xx}\mathscr{L}\, d = 2t^2 > 0$. So the sufficient condition holds, and $\hat{x}$ is a strict local minimizer.

1. **(G)** min $x_1 + x_2$ s.t. $x_1 x_2 = 1$, at $\hat{x} = (-1,-1)$ with $z = 1$.
2. **(G)** min $x_1^2 + x_2^2$ s.t. $x_1^2 + \frac{1}{4}x_2^2 \ge 1$, at $\hat{x} = (0,2)$ with $y = 4$.
3. **(G)** min $x_1^2 + 4x_1x_2 + x_2^2$ s.t. $x_1 \ge 0$, $x_2 \ge 0$, at $\hat{x} = (0,0)$ with $y = (0,0)$.
4. **(G)** min $x_1 + x_2^2$ s.t. $x_2 \ge 0$, $x_1^2 + x_2^2 = 1$, at $\hat{x} = (1,0)$ with $y = 0$ and $z = -\frac{1}{2}$.