

1 Certificates for nonlinear systems

Write every constraint in the form $f_i(x) \leq 0$, and write an equality $h(x) = 0$ as the two inequalities $h(x) \leq 0$ and $-h(x) \leq 0$. Every multiplier is nonnegative. For each system, do one of the following.

1. Find a feasible point.
2. Prove infeasibility by stating an infeasibility certificate.

Worked example 1.1

$x^2 \leq 1$, $x \geq 2$, with $x \in \mathbb{R}$.

Solution. The constraints are $x^2 - 1 \leq 0$ and $2 - x \leq 0$, with multipliers $y_1, y_2 \geq 0$. The Lagrangian is

$$\mathcal{L}(x, y) = y_1(x^2 - 1) + y_2(2 - x).$$

For $y_1 > 0$ it is a convex quadratic in x , minimized where $2y_1x - y_2 = 0$, at $x = \frac{y_2}{2y_1}$. For $y_1 = 0$ it is linear in x . So the dual function is

$$g(y) = \inf_x \mathcal{L}(x, y) = \begin{cases} -\frac{y_2^2}{4y_1} - y_1 + 2y_2 & \text{if } y_1 > 0, \\ -\infty & \text{if } y_1 = 0, y_2 > 0, \\ 0 & \text{if } y_1 = y_2 = 0. \end{cases}$$

To solve the dual problem, set $\frac{\partial g}{\partial y_2} = -\frac{y_2}{2y_1} + 2 = 0$, which gives $y_2 = 4y_1$. Then $g(y_1, 4y_1) = -4y_1 - y_1 + 8y_1 = 3y_1$. Letting $y_1 \rightarrow \infty$ gives $\sup_{y \geq 0} g(y) = +\infty > 0$, so the system is infeasible. A certificate is $y = (1, 4)$, for which

$$\mathcal{L}(x, y) = x^2 - 4x + 7 = (x - 2)^2 + 3 > 0 \quad \text{for all } x.$$

1. $x_1^2 + x_2^2 \leq 1$, $x_1 \geq 1$, $x_2 \geq 1$.
2. $x_1^2 + x_2^2 \leq 4$, $x_1 + x_2 \leq 3$.
3. $x^2 = 1$, $x \geq 2$, with $x \in \mathbb{R}$.
4. $x_1^2 + x_2^2 = 1$, $x_1 + x_2 \geq 2$.
5. $x^2 = 1$, $x^2 = 4$, with $x \in \mathbb{R}$.
6. $x_1x_2 = 1$, $x_1 + x_2 \leq 1$.

2 Certificates for linear systems

Farkas' lemma. Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Exactly one of the following holds.

- (i) There is $x \in \mathbb{R}^n$ such that $Ax \leq b$.
- (ii) There is $y \in \mathbb{R}^m$ such that $y \geq 0$, $A^T y = 0$, and $b^T y < 0$.

Write each system in the form $Ax \leq b$, and write an equality as two inequalities. Then use Farkas' lemma to do one of the following.

1. Find a feasible point, that is, x with $Ax \leq b$.
2. Prove infeasibility by stating a certificate, that is, $y \geq 0$ with $A^T y = 0$ and $b^T y < 0$.

Worked example 2.1

$x_1 - x_2 = 1$, $x_2 - x_1 \geq 1$.

Solution. The equality $x_1 - x_2 = 1$ is the two inequalities $x_1 - x_2 \leq 1$ and $-x_1 + x_2 \leq -1$. The inequality $x_2 - x_1 \geq 1$ is $x_1 - x_2 \leq -1$. So the system is $Ax \leq b$ with

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

For $y \geq 0$,

$$A^T y = \begin{bmatrix} y_1 - y_2 + y_3 \\ -y_1 + y_2 - y_3 \end{bmatrix}, \quad b^T y = y_1 - y_2 - y_3.$$

So $A^T y = 0$ exactly when $y_2 = y_1 + y_3$, and then $b^T y = -2y_3$. Any $y_3 > 0$ gives $b^T y < 0$, so (ii) holds and the system is infeasible. A certificate is $y = (0, 1, 1)$, with $A^T y = 0$ and $b^T y = -2 < 0$.

1. $x_1 + 3x_2 = 1$, $x_1 + x_2 = 3$.
2. $x_1 + x_2 \leq 1$, $x_1 \geq 2$, $x_2 \geq 0$.
3. $x_1 + x_2 = 1$, $2x_1 + 2x_2 = 3$, with $x \in \mathbb{R}^2$.
4. $x_1 + x_2 + x_3 = 1$, $x_1 - x_3 = 2$.
5. $x_1 + 2x_2 \leq 2$, $3x_1 + x_2 \leq 3$, $x_1 + x_2 \geq 2$.
6. $2x_1 + x_2 \geq 3$, $x_1 + 3x_2 \geq 4$, $x_1 + x_2 \leq 2$.

3 Lagrangian lower bounds

Write every constraint in the form $f_i(x) \leq 0$, and write an equality $h(x) = 0$ as the two inequalities $h(x) \leq 0$ and $-h(x) \leq 0$. Every multiplier is nonnegative. For each problem, complete the following steps.

1. Write the Lagrangian.
2. Compute the dual function.
3. Compute the optimal value of the dual problem.
4. Verify whether strong duality holds via Slater's condition.

Note. Slater's condition is sufficient for strong duality, but not necessary. Strong duality can still hold when Slater's condition fails.

Worked example 3.1

$\min x_1 + x_2$ s.t. $x_1^2 + x_2^2 \leq 2$.

Solution. The constraint is $x_1^2 + x_2^2 - 2 \leq 0$, with multiplier $y \geq 0$. The Lagrangian is

$$\mathcal{L}(x, y) = x_1 + x_2 + y(x_1^2 + x_2^2 - 2).$$

At $y = 0$ it is linear and unbounded below, so $g(0) = -\infty$. For $y > 0$, completing squares gives

$$\mathcal{L}(x, y) = y\left(x_1 + \frac{1}{2y}\right)^2 + y\left(x_2 + \frac{1}{2y}\right)^2 - \frac{1}{2y} - 2y,$$

so the dual function is $g(y) = -\frac{1}{2y} - 2y$ for $y > 0$ and $g(0) = -\infty$. The dual problem maximizes g over $y \geq 0$. The derivative $\frac{1}{2y^2} - 2$ is positive for $y < \frac{1}{2}$ and negative for $y > \frac{1}{2}$, so the optimal value is $d^* = g(\frac{1}{2}) = -2$.

The objective and the constraint function are convex, and $x = 0$ gives $0 - 2 < 0$, so Slater's condition holds. Hence strong duality holds, and $p^* = d^* = -2$.

1. $\min x_1 + 2x_2$ s.t. $x_1^2 + x_2^2 \leq 1$.
2. $\min x_1^2 + 3x_2^2$ s.t. $x_1 + x_2 \geq 4$.
3. $\min e^x$ s.t. $x \leq -1$, with $x \in \mathbb{R}$.
4. $\min x^2$ s.t. $x^2 \geq 1$, with $x \in \mathbb{R}$.
5. $\min x_1 + x_2$ s.t. $x_1 + 2x_2 \geq 2$, $x \geq 0$.
6. (G) $\min 2x_1^2 + 2x_1x_2 + 2x_2^2$ s.t. $x_1^2 + x_2^2 = 1$.
7. (G) $\min 2x_1x_2$ s.t. $x_1^2 = 1$, $x_2^2 = 1$.
8. (G) $\min x$ s.t. $(x - 1)^2 \leq 0$, with $x \in \mathbb{R}$.

4 Solving and certifying global optimality

Write every constraint in the form $f_i(x) \leq 0$, and write an equality $h(x) = 0$ as the two inequalities $h(x) \leq 0$ and $-h(x) \leq 0$. Every multiplier is nonnegative. For each problem, complete the following steps.

1. Find a global minimizer x^* by hand.
2. Write the Lagrangian.
3. Certify that x^* is globally optimal by finding multipliers y whose dual function value is $g(y) = f_0(x^*)$.
4. Verify whether strong duality holds via Slater's condition.

Note. Slater's condition is sufficient for strong duality, but not necessary. Strong duality can still hold when Slater's condition fails.

Worked example 4.1

$\min x_1^2 + x_2^2$ s.t. $x_1 + x_2 \geq 2$.

Solution. The objective is the squared distance to the origin. The closest point of the half-plane lies on the line $x_1 + x_2 = 2$, at $x^* = (1, 1)$, with $f_0(x^*) = 2$.

The constraint is $2 - x_1 - x_2 \leq 0$, with multiplier $y \geq 0$. The Lagrangian is

$$\mathcal{L}(x, y) = x_1^2 + x_2^2 + y(2 - x_1 - x_2).$$

For every feasible x and every $y \geq 0$, each term $y_i f_i(x)$ is at most zero, so $g(y) \leq \mathcal{L}(x, y) \leq f_0(x)$. Hence $g(y) = f_0(x^*)$ shows that no feasible point does better than x^* .

To find y , put weight zero on every inequality that is slack at x^* . Then $\mathcal{L}(x^*, y) = f_0(x^*)$, and $g(y) = f_0(x^*)$ exactly when x^* minimizes $\mathcal{L}(\cdot, y)$. Here the constraint is tight. The gradient of $\mathcal{L}$ at x^* is $(2 - y, 2 - y)$, which vanishes at $y = 2$. Then

$$\mathcal{L}(x, 2) = (x_1 - 1)^2 + (x_2 - 1)^2 + 2 \geq 2,$$

so $g(2) = 2 = f_0(x^*)$.

The objective and the constraint function are convex, and $x = (2, 2)$ gives $2 - 4 < 0$, so Slater's condition holds. Hence strong duality holds.

1. (G) $\min (x_1 - 2)^2 + x_2^2$ s.t. $x_1 + x_2 \leq 1$.
2. (G) $\min x_1^2 + 4x_1x_2 + x_2^2$ s.t. $x_1^2 + x_2^2 = 1$.
3. (G) $\min -x_1 - 2x_2$ s.t. $x_1 + x_2 \leq 4$, $x_1 \leq 3$, $x_2 \leq 3$, $x \geq 0$.
4. (G) $\min x_1^2 + x_2^2$ s.t. $x_1 + 2x_2 = 5$.
5. (G) $\min 2x_1x_2 + 2x_2x_3$ s.t. $x_i^2 = 1$ for $i = 1, 2, 3$.