# 1 Certificates for nonlinear systems

Write every constraint in the form $f_i(x) \le 0$, and write an equality $h(x) = 0$ as the two inequalities $h(x) \le 0$ and $-h(x) \le 0$. Every multiplier is nonnegative. For each system, do one of the following.

1. Find a feasible point.
2. Prove infeasibility by stating an infeasibility certificate.

**Worked example 1.1**

$x^2 \le 1$, $x \ge 2$, with $x \in \mathbb{R}$.

*Solution.* The constraints are $x^2 - 1 \le 0$ and $2 - x \le 0$, with multipliers $y_1, y_2 \ge 0$. The Lagrangian is

$$\mathscr{L}(x, y) = y_1(x^2 - 1) + y_2(2 - x).$$

For $y_1 > 0$ it is a convex quadratic in $x$, minimized where $2y_1x - y_2 = 0$, at $x = \frac{y_2}{2y_1}$. For $y_1 = 0$ it is linear in $x$. So the dual function is

$$g(y) = \inf_x \mathscr{L}(x, y) = \begin{cases} -\dfrac{y_2^2}{4y_1} - y_1 + 2y_2 & \text{if } y_1 > 0, \\ -\infty & \text{if } y_1 = 0,\ y_2 > 0, \\ 0 & \text{if } y_1 = y_2 = 0. \end{cases}$$

To solve the dual problem, set $\frac{\partial g}{\partial y_2} = -\frac{y_2}{2y_1} + 2 = 0$, which gives $y_2 = 4y_1$. Then $g(y_1, 4y_1) = -4y_1 - y_1 + 8y_1 = 3y_1$. Letting $y_1 \to \infty$ gives $\sup_{y \ge 0} g(y) = +\infty > 0$, so the system is infeasible. A certificate is $y = (1, 4)$, for which

$$\mathscr{L}(x, y) = x^2 - 4x + 7 = (x - 2)^2 + 3 > 0 \qquad \text{for all } x.$$

## Question 1

$x_1^2 + x_2^2 \le 1$, $x_1 \ge 1$, $x_2 \ge 1$.

**Solution**

The constraints are $x_1^2 + x_2^2 - 1 \le 0$, $1 - x_1 \le 0$, and $1 - x_2 \le 0$, with multipliers $y_1, y_2, y_3 \ge 0$. The Lagrangian is

$$\mathscr{L}(x, y) = y_1(x_1^2 + x_2^2 - 1) + y_2(1 - x_1) + y_3(1 - x_2).$$

For $y_1 > 0$ it is a convex quadratic in $x$, minimized at $x = \frac{1}{2y_1}(y_2, y_3)$. For $y_1 = 0$ it is linear in $x$. So

$$g(y) = \begin{cases} -\dfrac{y_2^2 + y_3^2}{4y_1} - y_1 + y_2 + y_3 & \text{if } y_1 > 0, \\ -\infty & \text{if } y_1 = 0,\ (y_2, y_3) \ne 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Setting $\frac{\partial g}{\partial y_2} = 0$ and $\frac{\partial g}{\partial y_3} = 0$ gives $y_2 = y_3 = 2y_1$. Then $g(y_1, 2y_1, 2y_1) = -2y_1 - y_1 + 4y_1 = y_1$. Letting $y_1 \to \infty$ gives $\sup_{y \ge 0} g(y) = +\infty > 0$, so the system is infeasible. A certificate is $y = (1, 2, 2)$, for which

$$\mathscr{L}(x, y) = (x_1 - 1)^2 + (x_2 - 1)^2 + 1 > 0 \qquad \text{for all } x.$$

## Question 2

$x_1^2 + x_2^2 \leq 4$, $x_1 + x_2 \leq 3$.

**Solution**

The point $x = (0, 0)$ gives $0 \leq 4$ and $0 \leq 3$, so the system is feasible.

## Question 3

$x^2 = 1$, $x \geq 2$, with $x \in \mathbb{R}$.

**Solution**

The equality is two inequalities, so the constraints are $x^2 - 1 \leq 0$, $1 - x^2 \leq 0$, and $2 - x \leq 0$, with multipliers $y_1, y_2, y_3 \geq 0$. The Lagrangian is

$$\mathscr{L}(x, y) = y_1(x^2 - 1) + y_2(1 - x^2) + y_3(2 - x) = (y_1 - y_2)\, x^2 - y_3 x - y_1 + y_2 + 2y_3.$$

For $y_1 > y_2$ it is a convex quadratic in $x$, minimized at $x = \frac{y_3}{2(y_1 - y_2)}$. For $y_1 < y_2$ it tends to $-\infty$ as $|x| \to \infty$. For $y_1 = y_2$ it is linear in $x$. So

$$g(y) = \begin{cases} -\dfrac{y_3^2}{4(y_1 - y_2)} - y_1 + y_2 + 2y_3 & \text{if } y_1 > y_2, \\ -\infty & \text{if } y_1 < y_2, \text{ or } y_1 = y_2 \text{ and } y_3 > 0, \\ 0 & \text{if } y_1 = y_2 \text{ and } y_3 = 0. \end{cases}$$

To solve the dual problem, set $y_2 = 0$. Then $\frac{\partial g}{\partial y_3} = -\frac{y_3}{2y_1} + 2 = 0$ gives $y_3 = 4y_1$, and $g(y_1, 0, 4y_1) = -4y_1 - y_1 + 8y_1 = 3y_1$. Letting $y_1 \to \infty$ gives $\sup_{y \geq 0} g(y) = +\infty > 0$, so the system is infeasible. A certificate is $y = (1, 0, 4)$, for which

$$\mathscr{L}(x, y) = (x - 2)^2 + 3 > 0 \qquad \text{for all } x.$$

## Question 4

$x_1^2 + x_2^2 = 1$, $x_1 + x_2 \geq 2$.

**Solution**

The equality is two inequalities, so the constraints are $x_1^2 + x_2^2 - 1 \leq 0$, $1 - x_1^2 - x_2^2 \leq 0$, and $2 - x_1 - x_2 \leq 0$, with multipliers $y_1, y_2, y_3 \geq 0$. The Lagrangian is

$$\begin{aligned} \mathscr{L}(x, y) &= y_1(x_1^2 + x_2^2 - 1) + y_2(1 - x_1^2 - x_2^2) + y_3(2 - x_1 - x_2) \\ &= (y_1 - y_2)(x_1^2 + x_2^2) - y_3(x_1 + x_2) - y_1 + y_2 + 2y_3. \end{aligned}$$

For $y_1 > y_2$ it is a convex quadratic in $x$, minimized at $x = \frac{y_3}{2(y_1 - y_2)}(1, 1)$. For $y_1 < y_2$ it tends to $-\infty$ as $\|x\| \to \infty$. For $y_1 = y_2$ it is linear in $x$. So

$$g(y) = \begin{cases} -\dfrac{y_3^2}{2(y_1 - y_2)} - y_1 + y_2 + 2y_3 & \text{if } y_1 > y_2, \\ -\infty & \text{if } y_1 < y_2, \text{ or } y_1 = y_2 \text{ and } y_3 > 0, \\ 0 & \text{if } y_1 = y_2 \text{ and } y_3 = 0. \end{cases}$$

To solve the dual problem, set $y_2 = 0$. Then $\frac{\partial g}{\partial y_3} = -\frac{y_3}{y_1} + 2 = 0$ gives $y_3 = 2y_1$, and $g(y_1, 0, 2y_1) = -2y_1 - y_1 + 4y_1 = y_1$. Letting $y_1 \to \infty$ gives $\sup_{y \geq 0} g(y) = +\infty > 0$, so the system is infeasible.

A certificate is $y = (1, 0, 2)$, for which

$$\mathscr{L}(x, y) = (x_1 - 1)^2 + (x_2 - 1)^2 + 1 > 0 \qquad \text{for all } x.$$

## Question 5

$x^2 = 1$, $x^2 = 4$, with $x \in \mathbb{R}$.

**Solution**

Each equality is two inequalities, so the constraints are $x^2 - 1 \le 0$, $1 - x^2 \le 0$, $x^2 - 4 \le 0$, and $4 - x^2 \le 0$, with multipliers $y_1, \dots, y_4 \ge 0$. The Lagrangian is

$$\begin{aligned}\mathscr{L}(x, y) &= y_1(x^2 - 1) + y_2(1 - x^2) + y_3(x^2 - 4) + y_4(4 - x^2) \\ &= (y_1 - y_2 + y_3 - y_4)\, x^2 - y_1 + y_2 - 4y_3 + 4y_4.\end{aligned}$$

It is bounded below in $x$ exactly when the coefficient of $x^2$ is nonnegative, and then its infimum is at $x = 0$. So

$$g(y) = \begin{cases} -y_1 + y_2 - 4y_3 + 4y_4 & \text{if } y_1 - y_2 + y_3 - y_4 \ge 0, \\ -\infty & \text{otherwise.} \end{cases}$$

To solve the dual problem, set $y_2 = y_3 = 0$. Then $g(y) = -y_1 + 4y_4$ with $y_1 \ge y_4$. For fixed $y_4$, $g$ is largest at $y_1 = y_4$, which gives $g = 3y_4$. Letting $y_4 \to \infty$ gives $\sup_{y \ge 0} g(y) = +\infty > 0$, so the system is infeasible. A certificate is $y = (1, 0, 0, 1)$, for which

$$\mathscr{L}(x, y) = (x^2 - 1) + (4 - x^2) = 3 > 0 \qquad \text{for all } x.$$

## Question 6

$x_1 x_2 = 1$, $x_1 + x_2 \le 1$.

**Solution**

The point $x = (-1, -1)$ gives $x_1 x_2 = 1$ and $x_1 + x_2 = -2 \le 1$, so the system is feasible.

## 2 Certificates for linear systems

**Farkas' lemma.** Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Exactly one of the following holds.

(i) There is $x \in \mathbb{R}^n$ such that $Ax \le b$.

(ii) There is $y \in \mathbb{R}^m$ such that $y \ge 0$, $A^T y = 0$, and $b^T y < 0$.

Write each system in the form $Ax \le b$, and write an equality as two inequalities. Then use Farkas' lemma to do one of the following.

1. Find a feasible point, that is, $x$ with $Ax \le b$.
2. Prove infeasibility by stating a certificate, that is, $y \ge 0$ with $A^T y = 0$ and $b^T y < 0$.

**Worked example 2.1**

$x_1 - x_2 = 1$, $x_2 - x_1 \ge 1$.

*Solution.* The equality $x_1 - x_2 = 1$ is the two inequalities $x_1 - x_2 \le 1$ and $-x_1 + x_2 \le -1$. The inequality $x_2 - x_1 \ge 1$ is $x_1 - x_2 \le -1$. So the system is $Ax \le b$ with

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

For $y \ge 0$,

$$A^T y = \begin{bmatrix} y_1 - y_2 + y_3 \\ -y_1 + y_2 - y_3 \end{bmatrix}, \qquad b^T y = y_1 - y_2 - y_3.$$

So $A^T y = 0$ exactly when $y_2 = y_1 + y_3$, and then $b^T y = -2y_3$. Any $y_3 > 0$ gives $b^T y < 0$, so (ii) holds and the system is infeasible. A certificate is $y = (0, 1, 1)$, with $A^T y = 0$ and $b^T y = -2 < 0$.

### Question 1

$x_1 + 3x_2 = 1$, $x_1 + x_2 = 3$.

**Solution**

Each equality is two inequalities. So the system is $Ax \le b$ with

$$A = \begin{bmatrix} 1 & 3 \\ -1 & -3 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}.$$

Subtracting the equations gives $2x_2 = -2$, so $x_2 = -1$ and $x_1 = 4$. At $x = (4, -1)$, $Ax = (1, -1, 3, -3) \le b$. So (i) holds and the system is feasible.

### Question 2

$x_1 + x_2 \le 1$, $x_1 \ge 2$, $x_2 \ge 0$.

**Solution**

The constraints are $x_1 + x_2 \le 1$, $-x_1 \le -2$, and $-x_2 \le 0$. So the system is $Ax \le b$ with

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

For $y \geq 0$,

$$A^T y = \begin{bmatrix} y_1 - y_2 \\ y_1 - y_3 \end{bmatrix}, \qquad b^T y = y_1 - 2y_2.$$

So $A^T y = 0$ exactly when $y_1 = y_2 = y_3 = t$, and then $b^T y = -t < 0$ for $t > 0$. So (ii) holds and the system is infeasible. A certificate is $y = (1, 1, 1)$, with $A^T y = 0$ and $b^T y = -1 < 0$.

## Question 3

$x_1 + x_2 = 1$, $2x_1 + 2x_2 = 3$, with $x \in \mathbb{R}^2$.

**Solution**

Each equality is two inequalities. So the system is $Ax \leq b$ with

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 2 & 2 \\ -2 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}.$$

For $y \geq 0$, both entries of $A^T y$ equal $y_1 - y_2 + 2y_3 - 2y_4$, and $b^T y = y_1 - y_2 + 3y_3 - 3y_4$. Look for $y$ with $y_2 = y_3 = 0$. Then $A^T y = 0$ reads $y_1 = 2y_4$, and $b^T y = 2y_4 - 3y_4 = -y_4 < 0$ for $y_4 > 0$. So (ii) holds and the system is infeasible. A certificate is $y = (2, 0, 0, 1)$, with $A^T y = (2 - 2,\ 2 - 2) = 0$ and $b^T y = 2 - 3 = -1 < 0$.

## Question 4

$x_1 + x_2 + x_3 = 1$, $x_1 - x_3 = 2$.

**Solution**

Each equality is two inequalities. So the system is $Ax \leq b$ with

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}.$$

Try $x_3 = 0$. The equations give $x_1 = 2$ and then $x_2 = -1$. At $x = (2, -1, 0)$, $Ax = (1, -1, 2, -2) \leq b$. So (i) holds and the system is feasible.

## Question 5

$x_1 + 2x_2 \leq 2$, $3x_1 + x_2 \leq 3$, $x_1 + x_2 \geq 2$.

**Solution**

The third constraint is $-x_1 - x_2 \leq -2$. So the system is $Ax \leq b$ with

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -1 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}.$$

For $y \geq 0$,

$$A^T y = \begin{bmatrix} y_1 + 3y_2 - y_3 \\ 2y_1 + y_2 - y_3 \end{bmatrix}, \qquad b^T y = 2y_1 + 3y_2 - 2y_3.$$

So $A^T y = 0$ exactly when $y_3 = y_1 + 3y_2 = 2y_1 + y_2$, that is, $y_1 = 2y_2$ and $y_3 = 5y_2$. Then $b^T y = 4y_2 + 3y_2 - 10y_2 = -3y_2 < 0$ for $y_2 > 0$. So (ii) holds and the system is infeasible. A certificate is $y = (2, 1, 5)$, with $A^T y = (2 + 3 - 5,\ 4 + 1 - 5) = 0$ and $b^T y = -3 < 0$.

## Question 6

$2x_1 + x_2 \geq 3$, $x_1 + 3x_2 \geq 4$, $x_1 + x_2 \leq 2$.

**Solution**

The first two constraints are $-2x_1 - x_2 \leq -3$ and $-x_1 - 3x_2 \leq -4$. So the system is $Ax \leq b$ with

$$A = \begin{bmatrix} -2 & -1 \\ -1 & -3 \\ 1 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix}.$$

At $x = (1, 1)$, $Ax = (-3, -4, 2) \leq b$. So (i) holds and the system is feasible.

# 3 Lagrangian lower bounds

Write every constraint in the form $f_i(x) \le 0$, and write an equality $h(x) = 0$ as the two inequalities $h(x) \le 0$ and $-h(x) \le 0$. Every multiplier is nonnegative. For each problem, complete the following steps.

1. Write the Lagrangian.
2. Compute the dual function.
3. Compute the optimal value of the dual problem.
4. Verify whether strong duality holds via Slater's condition.

*Note.* Slater's condition is sufficient for strong duality, but not necessary. Strong duality can still hold when Slater's condition fails.

**Worked example 3.1**

min $x_1 + x_2$ s.t. $x_1^2 + x_2^2 \le 2$.
*Solution.* The constraint is $x_1^2 + x_2^2 - 2 \le 0$, with multiplier $y \ge 0$. The Lagrangian is

$$\mathscr{L}(x, y) = x_1 + x_2 + y(x_1^2 + x_2^2 - 2).$$

At $y = 0$ it is linear and unbounded below, so $g(0) = -\infty$. For $y > 0$, completing squares gives

$$\mathscr{L}(x, y) = y\Big(x_1 + \frac{1}{2y}\Big)^2 + y\Big(x_2 + \frac{1}{2y}\Big)^2 - \frac{1}{2y} - 2y,$$

so the dual function is $g(y) = -\frac{1}{2y} - 2y$ for $y > 0$ and $g(0) = -\infty$. The dual problem maximizes $g$ over $y \ge 0$. The derivative $\frac{1}{2y^2} - 2$ is positive for $y < \frac{1}{2}$ and negative for $y > \frac{1}{2}$, so the optimal value is $d^\star = g(\frac{1}{2}) = -2$.
The objective and the constraint function are convex, and $x = 0$ gives $0 - 2 < 0$, so Slater's condition holds. Hence strong duality holds, and $p^\star = d^\star = -2$.

## Question 1

min $x_1 + 2x_2$ s.t. $x_1^2 + x_2^2 \le 1$.

**Solution**

The constraint is $x_1^2 + x_2^2 - 1 \le 0$, with multiplier $y \ge 0$. The Lagrangian is

$$\mathscr{L}(x, y) = x_1 + 2x_2 + y(x_1^2 + x_2^2 - 1).$$

At $y = 0$ it is linear and unbounded below, so $g(0) = -\infty$. For $y > 0$,

$$\mathscr{L}(x, y) = y\Big(x_1 + \frac{1}{2y}\Big)^2 + y\Big(x_2 + \frac{1}{y}\Big)^2 - \frac{5}{4y} - y,$$

so $g(y) = -\frac{5}{4y} - y$. Its derivative $\frac{5}{4y^2} - 1$ vanishes at $y = \frac{\sqrt{5}}{2}$, is positive before, and negative after. Hence $d^\star = g\big(\frac{\sqrt{5}}{2}\big) = -\frac{\sqrt{5}}{2} - \frac{\sqrt{5}}{2} = -\sqrt{5}$.
The objective and the constraint function are convex, and $x = 0$ gives $0 - 1 < 0$, so Slater's condition holds. Hence strong duality holds, and $p^\star = d^\star = -\sqrt{5}$.

## Question 2

min $x_1^2 + 3x_2^2$ s.t. $x_1 + x_2 \ge 4$.

**Solution**

The constraint is $4 - x_1 - x_2 \le 0$, with multiplier $y \ge 0$. The Lagrangian is

$$\mathscr{L}(x, y) = x_1^2 + 3x_2^2 + y(4 - x_1 - x_2) = \left(x_1 - \frac{y}{2}\right)^2 + 3\left(x_2 - \frac{y}{6}\right)^2 + 4y - \frac{y^2}{3},$$

so $g(y) = 4y - \frac{y^2}{3}$ for every $y \ge 0$. Its derivative $4 - \frac{2y}{3}$ vanishes at $y = 6$, so $d^\star = g(6) = 24 - 12 = 12$.
The objective and the constraint function are convex, and $x = (4, 4)$ gives $4 - 8 < 0$, so Slater's condition holds. Hence strong duality holds, and $p^\star = d^\star = 12$.

## Question 3

min $e^x$ s.t. $x \le -1$, with $x \in \mathbb{R}$.

**Solution**

The constraint is $x + 1 \le 0$, with multiplier $y \ge 0$. The Lagrangian is

$$\mathscr{L}(x, y) = e^x + y(x + 1).$$

For $y > 0$, $e^x \to 0$ and $y(x+1) \to -\infty$ as $x \to -\infty$, so $g(y) = -\infty$. At $y = 0$, $g(0) = \inf_x e^x = 0$. Hence $d^\star = 0$.
The objective and the constraint function are convex, and $x = -2$ gives $-2 + 1 < 0$, so Slater's condition holds. Hence strong duality holds, and $p^\star = d^\star = 0$.

## Question 4

min $x^2$ s.t. $x^2 \ge 1$, with $x \in \mathbb{R}$.

**Solution**

The constraint is $1 - x^2 \le 0$, with multiplier $y \ge 0$. The Lagrangian is

$$\mathscr{L}(x, y) = x^2 + y(1 - x^2) = (1 - y)\, x^2 + y.$$

For $y > 1$ the coefficient of $x^2$ is negative, so $g(y) = -\infty$. For $0 \le y \le 1$ the infimum is at $x = 0$, so $g(y) = y$. Hence $d^\star = g(1) = 1$.
The constraint function $1 - x^2$ is not convex, so Slater's condition does not apply, and it does not decide whether strong duality holds.

## Question 5

min $x_1 + x_2$ s.t. $x_1 + 2x_2 \ge 2$, $x \ge 0$.

**Solution**

The constraints are $2 - x_1 - 2x_2 \le 0$, $-x_1 \le 0$, and $-x_2 \le 0$, with multipliers $y_1, y_2, y_3 \ge 0$. The Lagrangian is

$$\begin{aligned}\mathscr{L}(x, y) &= x_1 + x_2 + y_1(2 - x_1 - 2x_2) - y_2 x_1 - y_3 x_2 \\ &= (1 - y_1 - y_2)\, x_1 + (1 - 2y_1 - y_3)\, x_2 + 2y_1.\end{aligned}$$

A linear function of $x$ is bounded below only if its coefficients vanish, so

$$g(y) = \begin{cases} 2y_1 & \text{if } y_2 = 1 - y_1 \text{ and } y_3 = 1 - 2y_1, \\ -\infty & \text{otherwise.} \end{cases}$$

Since $y_2, y_3 \geq 0$, the dual problem maximizes $2y_1$ subject to $0 \leq y_1 \leq \frac{1}{2}$. The maximum is at $y = (\frac{1}{2}, \frac{1}{2}, 0)$. Hence $d^\star = 1$.
The objective and the constraint functions are affine, hence convex, and $x = (1, 1)$ gives $2-3 < 0$, $-1 < 0$, and $-1 < 0$, so Slater's condition holds. Hence strong duality holds, and $p^\star = d^\star = 1$.

## Question 6

**(G)** min $2x_1^2 + 2x_1x_2 + 2x_2^2$ s.t. $x_1^2 + x_2^2 = 1$.

**Solution**

The equality is two inequalities, so the constraints are $x_1^2 + x_2^2 - 1 \leq 0$ and $1 - x_1^2 - x_2^2 \leq 0$, with multipliers $y_1, y_2 \geq 0$. The Lagrangian is

$$\begin{aligned} \mathscr{L}(x, y) &= 2x_1^2 + 2x_1x_2 + 2x_2^2 + y_1(x_1^2 + x_2^2 - 1) + y_2(1 - x_1^2 - x_2^2) \\ &= x^T M x - y_1 + y_2, \qquad M = \begin{bmatrix} 2 + y_1 - y_2 & 1 \\ 1 & 2 + y_1 - y_2 \end{bmatrix}. \end{aligned}$$

If $M \succeq 0$, then $x^T M x \geq 0$ with equality at $x = 0$, so $g(y) = -y_1 + y_2$. Otherwise some $u$ has $u^T M u < 0$, and $\mathscr{L}(tu, y) \to -\infty$ as $t \to \infty$, so $g(y) = -\infty$. A symmetric $2 \times 2$ matrix is PSD exactly when its diagonal entries and its determinant are nonnegative. Here that means $2 + y_1 - y_2 \geq 0$ and $(2 + y_1 - y_2)^2 \geq 1$, that is, $y_2 - y_1 \leq 1$. So

$$g(y) = \begin{cases} y_2 - y_1 & \text{if } y_2 - y_1 \leq 1, \\ -\infty & \text{otherwise.} \end{cases}$$

Hence $g(y) \leq 1$, with equality at $y = (0, 1)$, and $d^\star = 1$.
The constraint function $1 - x_1^2 - x_2^2$ is not convex, and no point satisfies both constraints strictly, so Slater's condition does not apply, and it does not decide whether strong duality holds.

## Question 7

**(G)** min $2x_1x_2$ s.t. $x_1^2 = 1$, $x_2^2 = 1$.

**Solution**

Each equality is two inequalities, so the constraints are $x_1^2 - 1 \leq 0$, $1 - x_1^2 \leq 0$, $x_2^2 - 1 \leq 0$, and $1 - x_2^2 \leq 0$, with multipliers $y_1, \ldots, y_4 \geq 0$. The Lagrangian is

$$\begin{aligned} \mathscr{L}(x, y) &= 2x_1x_2 + y_1(x_1^2 - 1) + y_2(1 - x_1^2) + y_3(x_2^2 - 1) + y_4(1 - x_2^2) \\ &= x^T M x - M_{11} - M_{22}, \qquad M = \begin{bmatrix} y_1 - y_2 & 1 \\ 1 & y_3 - y_4 \end{bmatrix}. \end{aligned}$$

If $M \succeq 0$, then $\inf_x x^T M x = 0$, attained at $x = 0$. Otherwise some $u$ has $u^T M u < 0$, and $\mathscr{L}(tu, y) \to -\infty$ as $t \to \infty$. A symmetric $2 \times 2$ matrix is PSD exactly when its diagonal entries

and its determinant are nonnegative. So

$$g(y) = \begin{cases} -M_{11} - M_{22} & \text{if } M_{11} \geq 0,\ M_{22} \geq 0,\ M_{11}M_{22} \geq 1, \\ -\infty & \text{otherwise.} \end{cases}$$

Where $g$ is finite, $M_{11} + M_{22} \geq 2\sqrt{M_{11}M_{22}} \geq 2$, so $g(y) \leq -2$. The choice $y = (1, 0, 1, 0)$ gives $M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \succeq 0$ and $g(y) = -2$. Hence $d^\star = -2$.

The constraint functions $1 - x_i^2$ are not convex, so Slater's condition does not apply, and it does not decide whether strong duality holds.

## Question 8

**(G)** min $x$ s.t. $(x-1)^2 \leq 0$, with $x \in \mathbb{R}$.

**Solution**

The constraint is $(x-1)^2 \leq 0$, with multiplier $y \geq 0$. The Lagrangian is

$$\mathscr{L}(x, y) = x + y(x-1)^2.$$

At $y = 0$ it is linear and unbounded below, so $g(0) = -\infty$. For $y > 0$, completing the square gives

$$\mathscr{L}(x, y) = y\left(x - 1 + \frac{1}{2y}\right)^2 + 1 - \frac{1}{4y},$$

so $g(y) = 1 - \frac{1}{4y}$ for $y > 0$ and $g(0) = -\infty$. This increases toward 1 as $y \to \infty$ and never reaches it. Hence $d^\star = 1$, and no $y$ attains it.

The objective and the constraint function are convex. But $(x-1)^2 \geq 0$ for every $x$, so no point has $(x-1)^2 < 0$. Slater's condition fails despite convexity, and it does not decide whether strong duality holds.

# 4 Solving and certifying global optimality

Write every constraint in the form $f_i(x) \le 0$, and write an equality $h(x) = 0$ as the two inequalities $h(x) \le 0$ and $-h(x) \le 0$. Every multiplier is nonnegative. For each problem, complete the following steps.

1. Find a global minimizer $x^\star$ by hand.
2. Write the Lagrangian.
3. Certify that $x^\star$ is globally optimal by finding multipliers $y$ whose dual function value is $g(y) = f_0(x^\star)$.
4. Verify whether strong duality holds via Slater's condition.

*Note.* Slater's condition is sufficient for strong duality, but not necessary. Strong duality can still hold when Slater's condition fails.

**Worked example 4.1**

min $x_1^2 + x_2^2$ s.t. $x_1 + x_2 \ge 2$.

*Solution.* The objective is the squared distance to the origin. The closest point of the half-plane lies on the line $x_1 + x_2 = 2$, at $x^\star = (1, 1)$, with $f_0(x^\star) = 2$.

The constraint is $2 - x_1 - x_2 \le 0$, with multiplier $y \ge 0$. The Lagrangian is

$$\mathscr{L}(x, y) = x_1^2 + x_2^2 + y(2 - x_1 - x_2).$$

For every feasible $x$ and every $y \ge 0$, each term $y_i f_i(x)$ is at most zero, so $g(y) \le \mathscr{L}(x, y) \le f_0(x)$. Hence $g(y) = f_0(x^\star)$ shows that no feasible point does better than $x^\star$.

To find $y$, put weight zero on every inequality that is slack at $x^\star$. Then $\mathscr{L}(x^\star, y) = f_0(x^\star)$, and $g(y) = f_0(x^\star)$ exactly when $x^\star$ minimizes $\mathscr{L}(\cdot, y)$. Here the constraint is tight. The gradient of $\mathscr{L}$ at $x^\star$ is $(2 - y, 2 - y)$, which vanishes at $y = 2$. Then

$$\mathscr{L}(x, 2) = (x_1 - 1)^2 + (x_2 - 1)^2 + 2 \ge 2,$$

so $g(2) = 2 = f_0(x^\star)$.

The objective and the constraint function are convex, and $x = (2, 2)$ gives $2 - 4 < 0$, so Slater's condition holds. Hence strong duality holds.

## Question 1

**(G)** min $(x_1 - 2)^2 + x_2^2$ s.t. $x_1 + x_2 \le 1$.

**Solution**

The objective is the squared distance to $(2, 0)$, which violates the constraint. The closest feasible point is the projection of $(2, 0)$ onto the line $x_1 + x_2 = 1$, which is $x^\star = (2, 0) - \frac{1}{2}(1, 1) = (\frac{3}{2}, -\frac{1}{2})$, with $f_0(x^\star) = \frac{1}{2}$.

The constraint is $x_1 + x_2 - 1 \le 0$, with multiplier $y \ge 0$. The Lagrangian is

$$\mathscr{L}(x, y) = (x_1 - 2)^2 + x_2^2 + y(x_1 + x_2 - 1).$$

The constraint is tight at $x^\star$. The gradient of $\mathscr{L}$ at $x^\star$ is $(2(x_1 - 2) + y,\ 2x_2 + y) = (y - 1,\ y - 1)$, which vanishes at $y = 1$. Then

$$\mathscr{L}(x, 1) = \left(x_1 - \tfrac{3}{2}\right)^2 + \left(x_2 + \tfrac{1}{2}\right)^2 + \tfrac{1}{2} \ge \tfrac{1}{2},$$

so $g(1) = \frac{1}{2} = f_0(x^\star)$.

The objective and the constraint function are convex, and $x = (0, 0)$ gives $0 - 1 < 0$, so Slater's condition holds. Hence strong duality holds.

## Question 2

**(G)** min $x_1^2 + 4x_1x_2 + x_2^2$ s.t. $x_1^2 + x_2^2 = 1$.

**Solution**

On the feasible set $f_0 = 1 + 4x_1x_2$, and $2|x_1x_2| \le x_1^2 + x_2^2 = 1$, so $f_0 \ge -1$. Equality holds when $x_1 = -x_2$, for example at $x^\star = \frac{1}{\sqrt{2}}(1, -1)$. So $f_0(x^\star) = -1$.

The equality is two inequalities, so the constraints are $x_1^2 + x_2^2 - 1 \le 0$ and $1 - x_1^2 - x_2^2 \le 0$, with multipliers $y_1, y_2 \ge 0$. The Lagrangian is

$$\begin{aligned}\mathscr{L}(x, y) &= x_1^2 + 4x_1x_2 + x_2^2 + y_1(x_1^2 + x_2^2 - 1) + y_2(1 - x_1^2 - x_2^2) \\ &= (1 + y_1 - y_2)(x_1^2 + x_2^2) + 4x_1x_2 - y_1 + y_2.\end{aligned}$$

Both constraints are tight at $x^\star$. The gradient of $\mathscr{L}$ at $x^\star$ is

$$\big(2(1 + y_1 - y_2)x_1 + 4x_2,\ 4x_1 + 2(1 + y_1 - y_2)x_2\big) = \tfrac{1}{\sqrt{2}}\big(2(y_1 - y_2) - 2,\ 2 - 2(y_1 - y_2)\big),$$

which vanishes when $y_1 - y_2 = 1$, for example at $y = (1, 0)$. Then

$$\mathscr{L}(x, y) = 2x_1^2 + 4x_1x_2 + 2x_2^2 - 1 = 2(x_1 + x_2)^2 - 1 \ge -1,$$

so $g(1, 0) = -1 = f_0(x^\star)$.

The objective is not convex, since along $x = t(1, -1)$ it equals $-2t^2$. The constraint function $1 - x_1^2 - x_2^2$ is not convex either. So Slater's condition does not apply, and it does not decide whether strong duality holds.

## Question 3

**(G)** min $-x_1 - 2x_2$ s.t. $x_1 + x_2 \le 4$, $x_1 \le 3$, $x_2 \le 3$, $x \ge 0$.

**Solution**

The feasible set is the polygon with vertices $(0, 0)$, $(3, 0)$, $(3, 1)$, $(1, 3)$, $(0, 3)$, where $f_0$ takes the values $0$, $-3$, $-5$, $-7$, $-6$. A linear function attains its minimum over a bounded polygon at a vertex, so $x^\star = (1, 3)$ with $f_0(x^\star) = -7$.

The constraints are $x_1 + x_2 - 4 \le 0$, $x_1 - 3 \le 0$, $x_2 - 3 \le 0$, $-x_1 \le 0$, and $-x_2 \le 0$, with multipliers $y_1, \dots, y_5 \ge 0$. The Lagrangian is

$$\mathscr{L}(x, y) = -x_1 - 2x_2 + y_1(x_1 + x_2 - 4) + y_2(x_1 - 3) + y_3(x_2 - 3) - y_4x_1 - y_5x_2.$$

At $x^\star$ only the first and third constraints are tight, so $y_2 = y_4 = y_5 = 0$. Then

$$\mathscr{L}(x, y) = (y_1 - 1)\, x_1 + (y_1 + y_3 - 2)\, x_2 - 4y_1 - 3y_3.$$

A linear function of $x$ is bounded below only if its coefficients vanish, which gives $y_1 = 1$ and $y_3 = 1$. With $y = (1, 0, 1, 0, 0)$, $\mathscr{L}(x, y) = -7$ for every $x$, so $g(y) = -7 = f_0(x^\star)$.

The objective and the constraint functions are affine, hence convex, and $x = (1, 1)$ gives $2 - 4 < 0$, $1 - 3 < 0$, $1 - 3 < 0$, $-1 < 0$, and $-1 < 0$, so Slater's condition holds. Hence strong duality holds.

## Question 4

**(G)** min $x_1^2 + x_2^2$ s.t. $x_1 + 2x_2 = 5$.

**Solution**

Eliminate $x_1 = 5 - 2x_2$. Then

$$f_0 = (5 - 2x_2)^2 + x_2^2 = 5x_2^2 - 20x_2 + 25 = 5(x_2 - 2)^2 + 5,$$

which is smallest at $x_2 = 2$. So $x^\star = (1, 2)$ with $f_0(x^\star) = 5$.
The equality is two inequalities, so the constraints are $5 - x_1 - 2x_2 \le 0$ and $x_1 + 2x_2 - 5 \le 0$, with multipliers $y_1, y_2 \ge 0$. The Lagrangian is

$$\begin{aligned}\mathscr{L}(x, y) &= x_1^2 + x_2^2 + y_1(5 - x_1 - 2x_2) + y_2(x_1 + 2x_2 - 5) \\ &= x_1^2 + x_2^2 + (y_1 - y_2)(5 - x_1 - 2x_2).\end{aligned}$$

Both constraints are tight at $x^\star$. The gradient of $\mathscr{L}$ at $x^\star$ is

$$\big(2 - (y_1 - y_2),\ 4 - 2(y_1 - y_2)\big),$$

which vanishes when $y_1 - y_2 = 2$, for example at $y = (2, 0)$. Then

$$\mathscr{L}(x, y) = (x_1 - 1)^2 + (x_2 - 2)^2 + 5 \ge 5,$$

so $g(2, 0) = 5 = f_0(x^\star)$.
The objective is convex and both constraint functions are affine. Slater's condition requires strict inequality only for constraints that are not affine, so any feasible point satisfies it, for example $x = (5, 0)$. Hence strong duality holds.

## Question 5

**(G)** min $2x_1x_2 + 2x_2x_3$ s.t. $x_i^2 = 1$ for $i = 1, 2, 3$.

**Solution**

Each product $x_ix_j$ is $\pm 1$, so $f_0 \ge -4$. Equality holds when $x_2 = -x_1$ and $x_2 = -x_3$, for example at $x^\star = (1, -1, 1)$. So $f_0(x^\star) = -4$.
Each equality is two inequalities, so the constraints are $x_i^2 - 1 \le 0$ and $1 - x_i^2 \le 0$ for $i = 1, 2, 3$, with multipliers $y_1, \dots, y_6 \ge 0$. The Lagrangian is

$$\begin{aligned}\mathscr{L}(x, y) = 2x_1x_2 + 2x_2x_3 &+ y_1(x_1^2 - 1) + y_2(1 - x_1^2) + y_3(x_2^2 - 1) \\ &+ y_4(1 - x_2^2) + y_5(x_3^2 - 1) + y_6(1 - x_3^2).\end{aligned}$$

All six constraints are tight at $x^\star$. The gradient of $\mathscr{L}$ at $x^\star$ is

$$\begin{aligned}&\big(2(y_1 - y_2)x_1 + 2x_2,\ 2(y_3 - y_4)x_2 + 2x_1 + 2x_3,\ 2(y_5 - y_6)x_3 + 2x_2\big) \\ &\quad = \big(2(y_1 - y_2) - 2,\ 4 - 2(y_3 - y_4),\ 2(y_5 - y_6) - 2\big),\end{aligned}$$

which vanishes when $y_1 - y_2 = 1$, $y_3 - y_4 = 2$, and $y_5 - y_6 = 1$, for example at $y = (1, 0, 2, 0, 1, 0)$. Then

$$\mathscr{L}(x, y) = x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 - 4 = (x_1 + x_2)^2 + (x_2 + x_3)^2 - 4 \ge -4,$$

so $g(y) = -4 = f_0(x^\star)$.
The constraint functions $1 - x_i^2$ are not convex, so Slater's condition does not apply, and it does not decide whether strong duality holds.