UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Department of Electrical and Computer Engineering

ECE 486: Control Systems

Homework 8 Solutions

Solution 1

(i) Denote M_{ij} to be the ij-th element in matrix M.

$$(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} B^T_{ik}A^T_{kj} = (B^TA^T)_{ij}$$

Hence $(AB)^T = B^TA^T$

(ii) By (i),

$$(ABC)^T = ((AB)C)^T = C^T(AB)^T = C^T(A^TB^T) = C^TB^TA^T$$

(iii) Denote $B := A^{-1}$, i.e.,

$$AB = I$$
$$\Rightarrow I = (AB)^T = B^TA^T \Rightarrow B^T = (A^T)^{-1}$$

Therefore $(A^{-1})^T = (A^T)^{-1}$

(iv)

$$\begin{aligned}(I - TAT^{-1})(T(I-A)^{-1}T^{-1}) &= (TIT^{-1} - TAT^{-1})(T(I-A)^{-1}T^{-1}) \\ &= (T(I-A)T^{-1})(T(I-A)^{-1}T^{-1}) \\ &= T(I-A)(I-A)^{-1}T^{-1} \\ &= TT^{-1} \\ &= I\end{aligned}$$

Hence $(I - TAT^{-1})^T = T(I-A)^{-1}T^{-1}$

(v) It is trivially true for $k = 0$. Suppose it is true for $k = n$. Then

$$(TAT^{-1})^{k+1} = (TAT^{-1})^k(TAT^{-1}) = (TA^kT^{-1})(TAT^{-1}) = TA^{k+1}T^{-1}$$

By induction, it is true for all $k \in \mathbb{N}_{\geq 0}$

Solution 2

(i)

$$p(s) = \det(sI - A) = \begin{vmatrix} s & 1 & -2/3 \\ 1 & s+2 & -1 \\ 0 & 3 & s-1 \end{vmatrix} = s^3 + s^2 - 1$$

(ii)

$$G(s) = C(Is - A)^{-1}B = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s & 1 & -2/3 \\ 1 & s+2 & -1 \\ 0 & 3 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{2s^2 - 1}{s^3 + s^2 - 1}$$

(iii) By formula, the CCF realization will be:

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u, \quad y = \begin{pmatrix} -1 & 0 & 2 \end{pmatrix} x$$

Solution 3

(i)

$$\mathcal{C} = (b|Ab) = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \Rightarrow \det(\mathcal{C}) = -2 \neq 0$$

Hence the system is controllable.

(ii)

$$\mathcal{C} = (b|Ab|A^2b) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$$

As the first row of $\mathcal{C}$ is 0, it is not full rank, so the system is not controllable.

Solution 4

(i)

$$\dot{\bar{v}}_1 = \dot{v}_1 = -(v_1 - v_2) + f_1 = \bar{v}_2 + f_1$$
$$\dot{\bar{v}}_2 = \dot{v}_2 - \dot{v}_1 = -(v_2 - v_1) + f_2 + (v_1 - v_2) - f_1 = -2\bar{v}_2 + f_2 - f_1$$

(ii) The original system is

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

The new system is already shown in (i):

$$\begin{pmatrix} \dot{\bar{v}}_1 \\ \dot{\bar{v}}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

T is extracted from the given relation:

$$\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

It can be computed that $T^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Therefore easy to check that

$$TAT^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} = \bar{A},$$

$$TB = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \bar{B}$$