UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Department of Electrical and Computer Engineering

ECE 486: Control Systems

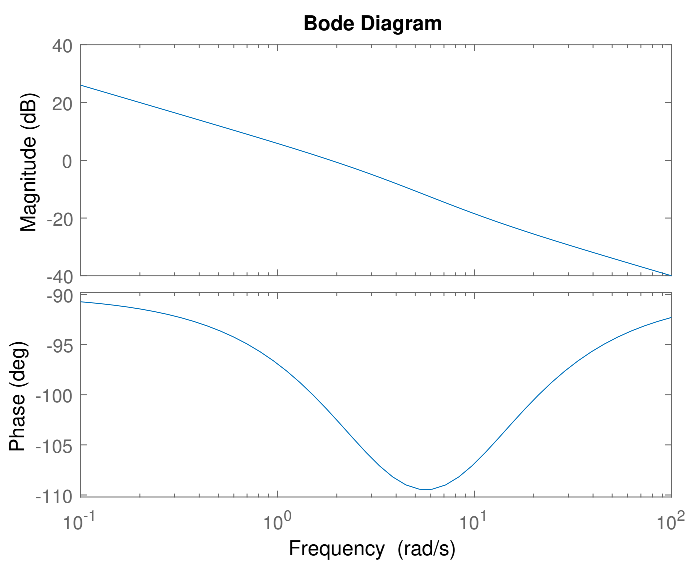
Homework 6 Solutions

Solution 1

a)

$$\text{Bode form: } KG(s) = 2\frac{\frac{s}{8}+1}{s\left(\frac{s}{4}+1\right)}$$

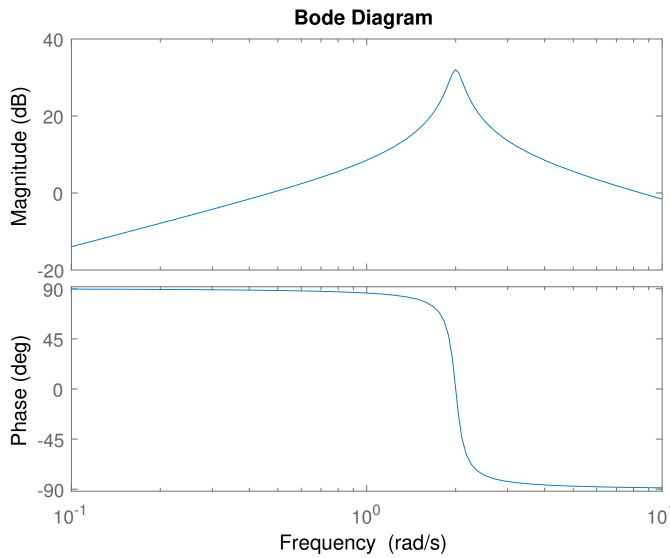
Break points: $\omega = 0$, $\omega = 4$, $\omega = 8$
$|KG(j1)| = 2$
Slope: -1 $\rightarrow$ -2 $\rightarrow$ -1
Phase: $-90^\circ \rightarrow -180^\circ \rightarrow -90^\circ$



b)

$$\text{Bode form: } KG(s) = 2\frac{s}{\left(\frac{s}{2}\right)^2 + 0.05s + 1}$$

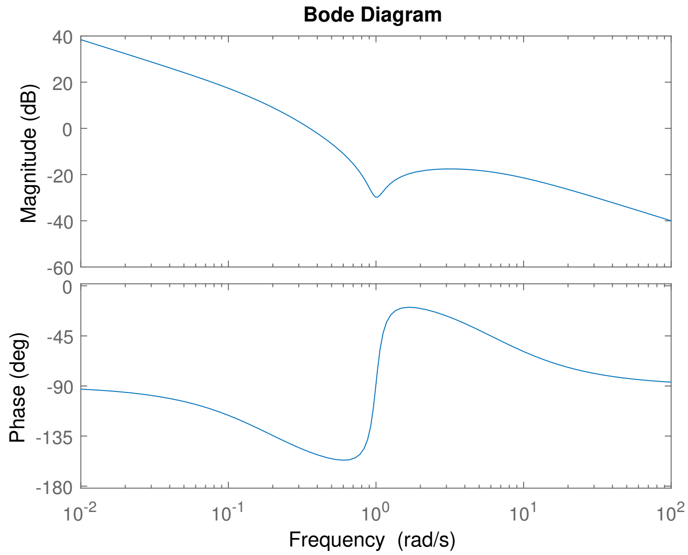
Break points: $\omega = 0$, $\omega = 2$
Damping ratio: $\zeta = 0.05$
$|KG(j0.1)| = 0.2$
Slope: +1 $\rightarrow$ -1
Phase: $+90^\circ \rightarrow -90^\circ$



c)

$$\text{Bode form: } KG(s) = \frac{1}{1.2} \frac{s^2 + 0.2s + 1}{s\left(\frac{s}{0.2} + 1\right)\left(\frac{s}{6} + 1\right)}$$

Break points: $\omega = 0.2$, $\omega = 1$, $\omega = 6$
Damping ratio: $\zeta = 0.1$
$|KG(j0.1)| = 8.3$
Slope: -1 $\rightarrow$ -2 $\rightarrow$ 0 $\rightarrow$ -1
Phase: $-90^\circ \rightarrow -180^\circ \rightarrow 0^\circ \rightarrow -90^\circ$



Solution 2

$$1 = |KG(j\omega_c)| = \frac{\omega_n^2}{\sqrt{(-\omega_c^2)^2 + (2\zeta\omega_n\omega_c)^2}}$$
$$\Rightarrow \omega_n^4 = \omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2$$

Let $x = \omega_c^2$, we have the quadratic equation

$$x^2 + 4\zeta^2\omega_n^2 x - \omega_n^4 = 0$$

and the roots are $x = \left(\pm\sqrt{4\zeta^4+1} - 2\zeta^2\right)\omega_n^2$. The negative root can be omitted since $x = \omega_c^2 \geq 0$. Hence

$$\omega_c = \left(\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}\right)\omega_n$$

Negative root is also omitted here since we always take positive frequency. Therefore

$$\begin{aligned}
\text{PM} &= \pi + \angle KG(j\omega_c) \\
&= \pi - \tan^{-1}\left(\frac{2\zeta\omega_n\omega_c}{-\omega_c^2}\right) \\
&= \tan^{-1}\left(\frac{2\zeta\omega_n}{\omega_c}\right) \\
&= \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right)
\end{aligned}$$

which is independent of ω_n.

Solution 3

(i) The characteristic equation is

$$s^3 + 4s^2 + 8s + K = 0$$

If there is an root on the $j\omega$-axis, then

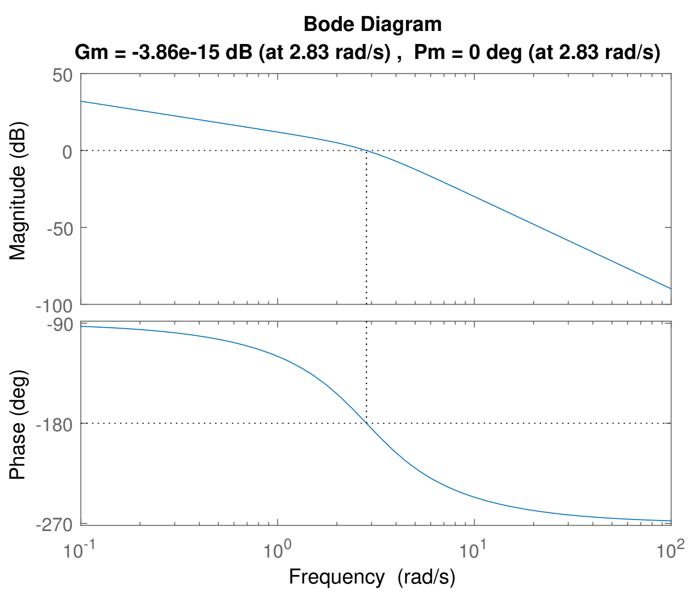
$$(j\omega)^3 + 4(j\omega)^2 + 8j\omega + K = 0$$

Inspecting on the real and imaginary parts,

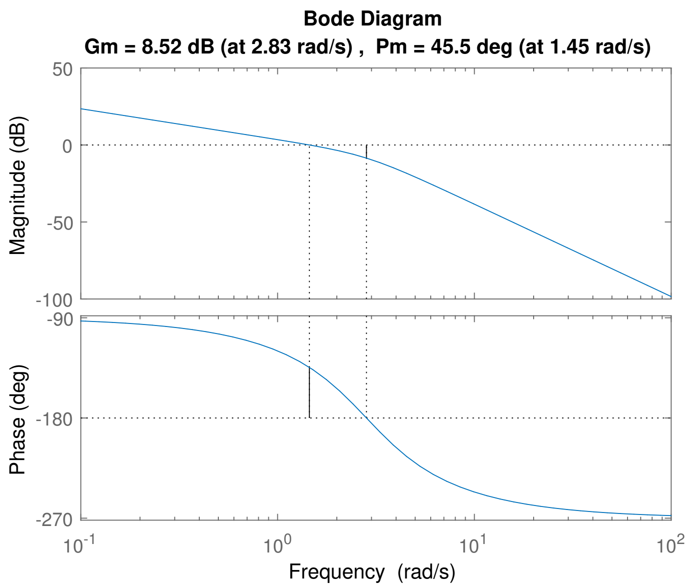
$$\begin{array}{r} -4\omega^2 + K = 0 \\ -\omega^3 + 8\omega = 0 \end{array} \Rightarrow \begin{array}{c} \omega = 2\sqrt{2} \\ K = 32 \end{array}$$

The other trivial solution is omitted here.

(ii) The MATLAB Bode plot suggests both phase margin and gain margin are 0, corresponding to the system being marginally stable. This is exactly the case when there are closed loop poles on the $j\omega$-axis.



(iii) The gain and phase margins can be measured from the Bode plot, or computed directly using "margin" commend.



In this case, $\text{PM} = 45.5°, \text{GM} = 8.52\text{dB}$.

(iv)

$$\text{PM} = 180° + \angle(KG(j\omega_c)) = 60°$$

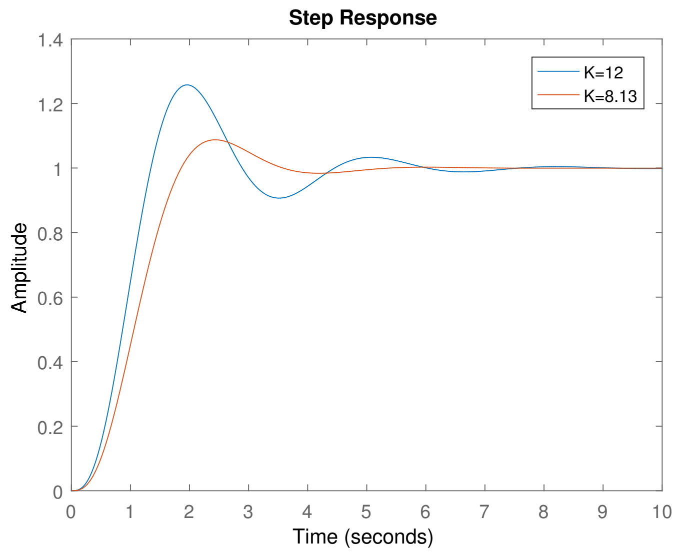
$$\Rightarrow -90° - \tan^{-1}\left(\frac{4\omega_c}{8-\omega_c^2}\right) = -120°$$

$$\Rightarrow \omega_c = 2\sqrt{5} - 2\sqrt{3}$$

Therefore

$$1 = |KG(j\omega_c)| \Rightarrow 1 = \frac{K}{\omega_c(\sqrt{(8-\omega_c^2)^2 + (4\omega_c)^2})} \Rightarrow K = 8.13$$

(v) System from (iv) has better damping. The larger phase margin is, the better damping and smaller overshoot the step response will have.



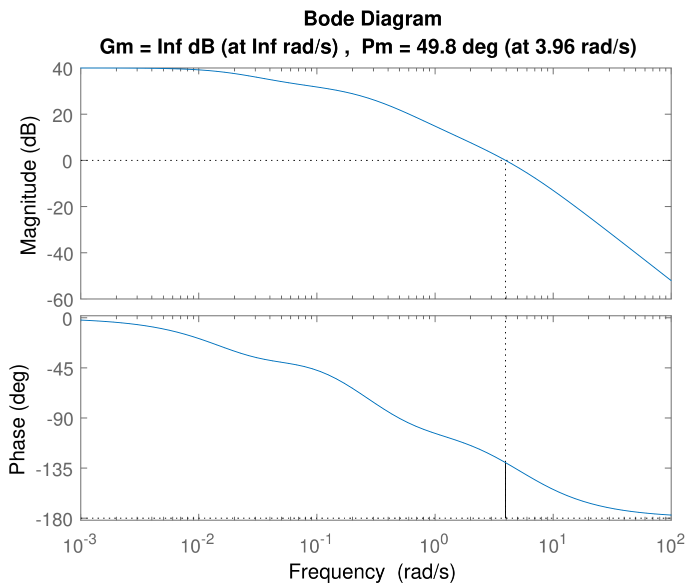
Solution 4

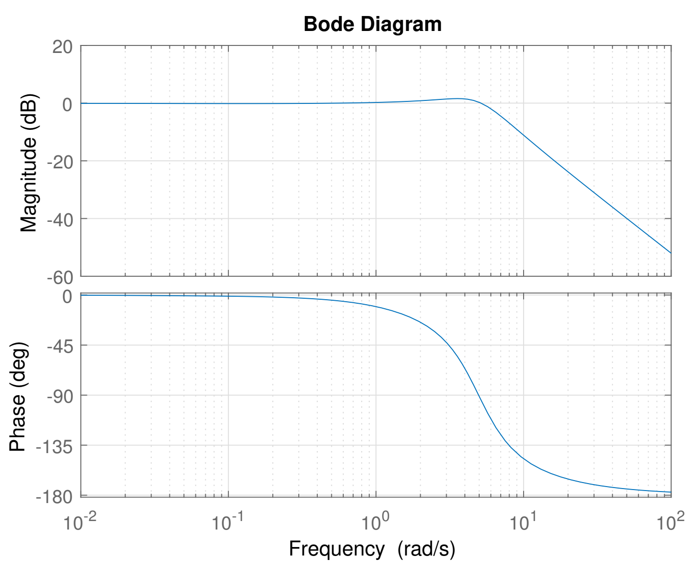
$$KD(s) = 4 \cdot \frac{\frac{s}{0.8}+1}{\frac{s}{5}+1} \cdot \frac{s+0.05}{s+0.02}.$$

(a) The attached Bode shows that PM $\approx 50^\circ$ with $\omega_c = 4$ so the phase-margin is NOT achieved. The error is

$$e(\infty) = \frac{1}{1+KD(0)G(0)} = \frac{1}{101} < 0.01$$

and the closed-loop bode shows that BW ≈ 6.36.





(b) To bring down BW (it's > 6), we need to reduce ω_c, also we need to increase our PM so we start with the lag controller from part 1(b) and pull the lead pole a little further to increase the phase which is provided by the lead controller and adjust the gain for error requirements:

$$KD(s) = 3 \cdot \frac{\frac{s}{0.8} + 1}{\frac{s}{8} + 1} \cdot \frac{s + 0.04}{s + 0.01}.$$

The attached Bodes show that PM $= 65°$ and BW $= 5.34$ and

$$e(\infty) = \left. \frac{1}{1 + KD(s)G(s)} \right|_{s=0} = \frac{1}{121} < 0.01.$$