UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Department of Electrical and Computer Engineering

ECE 486: Control Systems

Homework 4 Solutions

[redacted]

Solution 1

(i)

$$\begin{array}{cccc} s^4 & 1 & 15 & 1 \\ s^3 & 10 & 20 & \\ s^2 & 13 & 1 & \\ s^1 & \frac{250}{13} & & \\ s^0 & 1 & & \end{array}$$

No change of sign in the first column ⇒ No RHP roots.

(ii) There are negative coefficients ⇒ RHP roots exist.

(iii) There are negative coefficients ⇒ RHP roots exist.

(iv)

$$\begin{array}{cccc} s^4 & 1 & 12 & 1 \\ s^3 & 10 & 20 & \\ s^2 & 10 & 1 & \\ s^1 & 19 & & \\ s^0 & 1 & & \end{array}$$

No change of sign in the first column ⇒ No RHP roots.

Solution 2

The closed loop transfer function is:

$$G_{cl} = \frac{KG}{1+KG} = \frac{\frac{K}{s^3+3s^2+s+1}}{1+\frac{K}{s^3+3s^2+s+1}} = \frac{K}{s^3+3s^2+s+(K+1)}$$

Construct the Routh-Hurwitz array:

$$\begin{array}{ccc} s^3 & 1 & 1 \\ s^2 & 3 & (K+1) \\ s^1 & \frac{3-(K+1)}{3} & \\ s^0 & K+1 & \end{array}$$

Hence for the system to be stable, we need:

$$\left.\begin{array}{c}\frac{3-(K+1)}{3} > 0 \\ K+1 > 0\end{array}\right. \Rightarrow -1 < K < 2$$

In addition, the sytem is unstable when $K \geq 2$

Solution 3

(i) Constant reference, say unit step: $R(s) = \frac{1}{s}$. Assume there is no disturbance, i.e., $W = 0$. Then

$$Y = KGR = \frac{K}{s(s+p)}$$

Using Final Value Theorem,

$$y(\infty) = r(\infty) = 1 \Rightarrow 1 = \lim_{s \to 0} Ys = \lim_{s \to 0} \frac{K}{s+p} = \frac{K}{p} \Rightarrow K = p$$

(2) Constant disturbance, say unit step: $W(s) = \frac{1}{s}$. Assume there is no reference, i.e., $R = 0$. Then

$$\frac{Y}{W} = CKG = \frac{Cp}{s+p}$$

which means the DC gain from W to Y is C. Using Final Value Theorem,

$$y(\infty) = \lim_{s \to 0} Ys = \lim_{s \to 0} \frac{Cp}{s+p} = C \neq 0$$

Therefore the system is unable to reject constant disturbances.

Solution 4

(i) Recall $T_{r \to y} = \frac{KP}{1+KP}$. When $n > 0$,

$$0 \neq c = \lim_{s \to 0} \frac{1 - T_{r \to y}(s)}{s^n} = \lim_{s \to 0} \frac{\frac{1}{1+KP}}{s^n} = \lim_{s \to 0} \frac{1}{s^n + s^n KP} = \lim_{s \to 0} \frac{1}{s^n K(s)P(s)}$$

$$\Leftrightarrow \lim_{s \to 0} s^n K(s)P(s) = \frac{1}{c} \neq 0$$

When $n = 0$,

$$c = \lim_{s \to 0}(1 - T_{r \to y}(s)) = \lim_{s \to 0} \frac{1}{1+KP} \Rightarrow K(0)P(0) = \frac{1}{c} - 1 < \infty$$

Also notice that $K(0)P(0) \neq 0$, therefore

$$\lim_{s \to 0} s^n K(s)P(s) = \lim_{s \to 0} nK(s)P(s) = \frac{1}{c} - 1 \neq 0$$

Hence the system has type n.

(ii) Notice that signal from W to Y can be viewed as with open loop P and feedback K. Hence

$$T_{w\to y} = \frac{P}{1+KP}$$

(iii) Without loss of generality, we can always assume that $T_{w\to y}(s) = s^{k'}\frac{A(s)}{B(s)}$ with $k' \in \mathbb{N}_{\geq 0}$ and A, B polynomials with real coefficients such that $A(0) \neq 0$, $B(0) \neq 0$. Notice that

$$\lim_{s\to 0} \frac{T_{w\to y}(s)}{s^k} = \lim_{s\to 0} s^{(k'-k)}\frac{A(s)}{B(s)} = \frac{A(0)}{B(0)} \lim_{s\to 0} s^{(k'-k)}$$

If $k' > k$,

$$\lim_{s\to 0} \frac{T_{w\to y}(s)}{s^k} = 0$$

If $k' < k$, $\lim_{s\to 0} \frac{T_{w\to y}(s)}{s^k}$ is not defined. Hence we must have $k' = k$. In other words, $T_{w\to y}$ has type k with respect to disturbance inputs if it has a *zero* of order k at the origin.

(iv) Let $w(t)$ be a degree of m polynomial disturbances. Then $W(s) = \frac{W_0}{s^{m+1}}$. By Final Value Theorem,

$$y(\infty) = \lim_{s\to 0} T_{w\to y}(s)W(s)s = \lim_{s\to 0} s^{k-m}\frac{W_0 A(s)}{B(s)} = \begin{cases} 0 & \text{if } m < k \\ \frac{W_0 A(0)}{B(0)} & \text{if } m = k \\ \text{not defined} & \text{if } m > k \end{cases}$$

Hence the system of type k with respect to disturbances can achieve perfect steady-state rejection of polynomial disturbances of degree $m < k$, but not when mk.

(v) Recall $T_{w\to y} = \frac{P}{1+KP}$.

(a)

$$T_{\text{P}} = \frac{P}{1+K_P P} = \frac{\frac{1}{s^2+1}}{1+\frac{K_P}{s^2+1}} = \frac{1}{s^2+(K_P+1)}$$

no *zero* at origin, hence type 0.

(b)

$$T_{\text{PD}} = \frac{P}{1+(K_P+K_D s)P} = \frac{\frac{1}{s^2+1}}{1+\frac{(K_P+K_D s)}{s^2+1}} = \frac{1}{s^2+K_D s+(K_P+1)}$$

no *zero* at origin, hence type 0.

(c)

$$T_{\text{PID}} = \frac{P}{1+(K_P+K_D s+\frac{K_I}{s})P} = \frac{\frac{1}{s^2+1}}{1+\frac{(K_P s+K_D s^2+K_i)}{s(s^2+1)}} = \frac{s}{s^3+K_D s^2+(K_P+1)s+K_I}$$

a *zero* at origin, hence type 1.