UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Department of Electrical and Computer Engineering

ECE 486: Control Systems

Homework 2 Solutions

Spring 2024

Problem 1

(i) $f_1(t) = \sin(2t) = \dfrac{e^{2jt} - e^{-2jt}}{2j}$

$$
\begin{aligned}
U_1 = \mathcal{L}\{\sin(2t)\} &= \int_0^\infty \left(\frac{e^{2jt} - e^{-2jt}}{2j}\right) e^{-st}\, dt \\
&= \frac{1}{2j}\int_0^\infty (e^{-(s-2j)t} - e^{-(s+2j)t})\, dt \\
&= \frac{1}{2j}\left(-\frac{e^{-(s-2j)t}}{s-2j}\Big|_0^\infty + \frac{e^{-(s+2j)t}}{s+2j}\Big|_0^\infty \right) \\
&= \frac{1}{2j}\left(\frac{4j}{s^2+4}\right) \\
&= \frac{2}{s^2+4}
\end{aligned}
$$

$f_2(t) = e^{-3t}$

$$
\begin{aligned}
F_2 = \mathcal{L}\{e^{-3t}\} &= \int_0^\infty e^{-3t} e^{-st}\, dt \\
&= \int_0^\infty e^{-(s+3)t)}\, dt \\
&= -\frac{e^{-(s+3)t}}{s+3}\Big|_0^\infty \\
&= \frac{1}{s+3}
\end{aligned}
$$

$f_3(t) = \sin(2t) + e^{-3t}$

By the Linearity of Laplace Transform,

$$
\begin{aligned}
F_3 = \mathcal{L}\{\sin(2t) + e^{-3t}\} &= \mathcal{L}\{\sin(2t)\} + \mathcal{L}\{e^{-3t}\} \\
&= \frac{2}{s^2+4} + \frac{1}{s+3}
\end{aligned}
$$

(ii) *The Final Value Theorem : If all poles of $sY(s)$ are in the left half of the s-plane, then*

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

$$\lim_{t\to\infty} f_1(t) = \lim_{s\to 0} sF_1(s) = \lim_{s\to 0} \frac{2s}{s^2+4} = 0 \text{ (Invalid)}$$
$$\lim_{t\to\infty} f_2(t) = \lim_{s\to 0} sF_2(s) = \lim_{s\to 0} \frac{s}{s+3} = 0 \text{ (Valid)}$$
$$\lim_{t\to\infty} f_3(t) = \lim_{s\to 0} sF_3(s) = \lim_{s\to 0} \left(\frac{2s}{s^2+4} + \frac{s}{s+3}\right) = 0 \text{ (Invalid)}$$

Problem 2

Compute *by hand* the step responses of

(i) $H_1(s) = \dfrac{2}{s+4}$

$$Y_1(s) = \frac{1}{s}H_1(s) = \frac{2}{s(s+4)} = \frac{C_1}{s} + \frac{C_2}{s+4}$$
$$C_1 = \left.\frac{2}{s+4}\right|_{s=0} = \frac{1}{2}, \qquad C_2 = \left.\frac{2}{s}\right|_{s=-4} = -\frac{1}{2}$$

Hence,

$$Y_1(s) = \frac{1}{2s} - \frac{1}{2(s+4)}$$

Compute $y_1(t)$ by Reverse Laplace Transform

$$\mathcal{L}^{-1}\{Y_1(s)\} = y_1(t) = \frac{1}{2} - \frac{1}{2}e^{-4t}$$

Steady-state response: $\lim_{t\to\infty} y_1(t) = \frac{1}{2}$
DC Gain:

$$sY_1(s) = sH_1(s)\left.\frac{1}{s}\right|_{s=0} = \frac{1}{2}$$

Therefore, The steady-state response to the unit step function is equal to the DC gain.

(ii) $H_2(s) = \dfrac{2}{s-4}$

$$Y_2(s) = \frac{1}{s}H_2(s) = \frac{2}{s(s-4)} = \frac{C_1}{s} + \frac{C_2}{s-4}$$

$$C_1 = \frac{2}{s-4}\bigg|_{s=0} = -\frac{1}{2}, \qquad C_2 = \frac{2}{s}\bigg|_{s=4} = \frac{1}{2}$$

Hence,

$$Y_2(s) = -\frac{1}{2s} + \frac{1}{2(s-4)}$$

Compute $y_2(t)$ by Reverse Laplace Transform

$$\mathcal{L}^{-1}\{Y_2(s)\} = y_2(t) = -\frac{1}{2} + \frac{1}{2}e^{4t}$$

Steady-state response: $\lim_{t\to\infty} y_1(t) = \infty$
Since there is a pole in the RHP, therefore, FVT is invalid and DC gain cannot be determined.

Problem 3

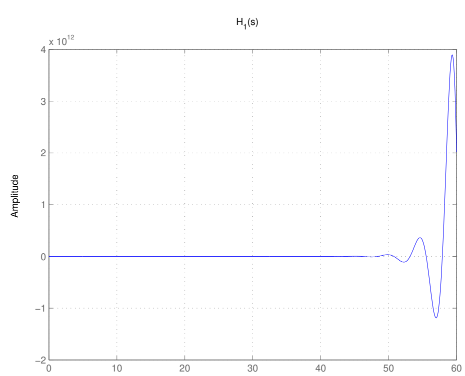
$$H_1(s) = \frac{1}{s^2 - s + 2}, \qquad H_2(s) = \frac{s-3}{s^2 + 5s + 6}$$

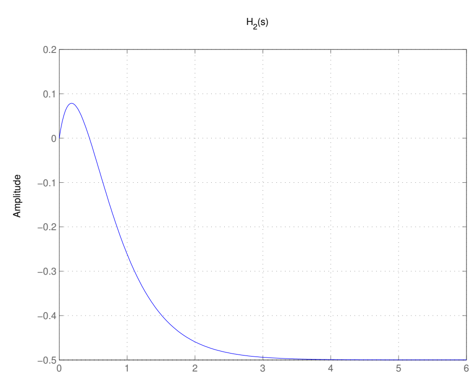
(i) Compute DC Gain

(a) FVT does not apply for the first case because the poles of $H_1(s)$ are not in an open LHP

(b) Poles of $H_2(s)$ are both in open LHP, hence, FVT applies.
DC Gain of $H_2(s) = sH_2(s)\frac{1}{s}\bigg|_{s=0} = -\frac{3}{6} = -\frac{1}{2}$

(ii) Step responses





(iii) For the first case, FVT is invalid because there exists a pole in the RHP causing the system to be unstable. The step response agrees because we can see from the plot that the step response does not converge.

As for the second system, we can see from the plot that the step response converges to -0.5, which agrees with the result from part (i).

Problem 4

(i) Rearrange the equation of motion:

$$\ddot{\theta} = -\frac{\gamma}{J}\dot{\theta} + \frac{mgl}{J}\sin\theta + \frac{l}{J}F\cos\theta.$$

Take $x_1 = \theta, x_2 = \dot{\theta}, u = F, y = x_1$ to get the nonlinear state-space model

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{\gamma}{J}x_2 + \frac{mgl}{J}\sin x_1 + \frac{l}{J}u\cos x_1\\ y &= x_1\end{aligned}$$

which corresponds to $f(x_1, x_2, u) = \begin{pmatrix} f_1(x_1,x_2,u) \\ f_2(x_1,x_2,u)\end{pmatrix}$ with

$$f_1(x_1,x_2,u) = x_2, \qquad f_2(x_1,x_2,u) = -\frac{\gamma}{J}x_2 + \frac{mgl}{J}\sin x_1 + \frac{l}{J}u\cos x_1.$$

(ii) To check the equilibrium condition, we substitute the zero-state/zero-input point into the state-space equation:

$$\begin{aligned}f_1(0,0,0) &= x_2\big|_{x_2=0} = 0\\ f_2(0,0,0) &= -\frac{\gamma}{J}x_2 + \frac{mgl}{J}\sin x_1 + \frac{l}{J}u\cos x_1\big|_{x_1=x_2=u=0} = 0\end{aligned}$$

To obtain the linearized model, we take the Jacobians of $f(x_1, x_2, u)$ with respect to x_1, x_2 and with respect to u and evaluate them at $x_1 = x_2 = u = 0$:

$$\begin{aligned}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} &= \begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{pmatrix}\Bigg|_{x_1=x_2=u=0}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}\frac{\partial f_1}{\partial u}\\ \frac{\partial f_2}{\partial u}\end{pmatrix}\Bigg|_{x_1=x_2=u=0} u\\ &= \begin{pmatrix}0 & 1\\ \frac{mgl}{J} & -\frac{\gamma}{J}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ \frac{l}{J}\end{pmatrix}u.\end{aligned}$$

The output is given by $y = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$.

(iii) The linearized state-space model from part (ii) is:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{mgl}{J}x_1 - \frac{\gamma}{J}x_2 + \frac{l}{J}u\\ y &= x_1\end{aligned}$$

We take the Laplace transform (assuming zero i.c.'s):

$$\begin{aligned}sX_1(s) &= X_2(s)\\ sX_2(s) &= \frac{mgl}{J}X_1(s) - \frac{\gamma}{J}X_2(s) + \frac{l}{J}U(s)\\ Y(s) &= X_1(s)\end{aligned}$$

Substituting sX_1 for X_2 in the second equation and rearranging, we have

$$\left(s^2 + \frac{\gamma}{J}s - \frac{mgl}{J}\right)X_1(s) = \frac{l}{J}U(s).$$

Since $Y(s) = X_1(s)$, the transfer function is given by

$$\begin{aligned}H(s) &= \frac{Y(s)}{U(s)}\\ &= \frac{\frac{l}{J}}{s^2 + \frac{\gamma}{J}s - \frac{mgl}{J}}\\ &= \frac{l}{Js^2 + \gamma s - mgl}\end{aligned}$$

Problem 5

<u>Observer Canonical Form:</u>

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & -a_0\\ 1 & -a_1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}b_0\\ b_1\end{pmatrix}u, \qquad y = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

The differential equations:

$$\dot{x_1} = -a_0x_2 + b_0u, \qquad y = x_2$$

$$\dot{x_2} = x_1 - a_1x_2 + b_1u$$

In s-domain:

$$sX_1 = -a_0X_2 + b_0U, \qquad Y = X_2$$

$$sX_2 = X_1 - a_1X_2 + b_1U$$

Multiply s to the sX_2 equation:

$$s^2X_2 = sX_1 - a_1sX_2 + b_1sU$$

Substitute sX_1,

$$s^2X_2 = -a_0X_2 + b_0U - a_1sX_2 + b_1sU$$
$$(s^2 + a_1s + a_0)X_2 = (b_0 + b_1s)U$$

Since $Y = X_2$, hence

$$(s^2 + a_1s + a_0)Y = (b_0 + b_1s)U$$

Transfer function, $H(s) = \dfrac{Y(s)}{U(s)} = \dfrac{b_1s + b_0}{s^2 + a_1s + a_0}$

Controller Canonical Form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} b_0 & b_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The differential equations:

$$\dot{x_1} = x_2 \qquad y = b_0x_1 + b_1x_2$$

$$\dot{x_2} = -a_0x_1 - a_1x_2 + u$$

In s-domain:

$$sX_1 = X_2 \qquad Y = b_0X_1 + b_1X_2$$
$$sX_2 = -a_0X_1 - a_1X_2 + U$$

Substitute $sX_1 = X_2$ in the Y equation,

$$Y = b_0X_1 + b_1sX_1$$
$$\therefore \frac{Y}{X_1} = b_0 + b_1s$$

Substitute $sX_1 = X_2$ in the sX_2 equation

$$s^2X_1 = -a_0X_1 - a_1sX_1 + U$$
$$\therefore \frac{X_1}{U} = \frac{1}{s^2 + a_1s + a_0}$$

Therefore,

$$\frac{Y}{X_1} \cdot \frac{X_1}{U} = (b_0 + b_1s) \cdot \frac{1}{s^2 + a_1s + a_0}$$
$$\frac{Y}{U} = \frac{b_1s + b_0}{s^2 + a_1s + a_0}$$

Transfer function, $H(s) = \dfrac{Y(s)}{U(s)} = \dfrac{b_1 s + b_0}{s^2 + a_1 s + a_0}$

This shows that the same transfer function can be realized by several different state-space models.