UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Department of Electrical and Computer Engineering

ECE 486: Control Systems

Homework 1 Solutions

Spring 2024

Solution 1

Solution:

(i)

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 4 \times 4 = \lambda^2 - 3\lambda - 14$$

$$P(\lambda) = 0 \Rightarrow \lambda_1 = \frac{3+\sqrt{65}}{2}, \lambda_2 = \frac{3-\sqrt{65}}{2}$$

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(ii)

$$P(\lambda) = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) + 3 \times 3 = \lambda^2 - 2\lambda + 10$$

$$P(\lambda) = 0 \Rightarrow \lambda_1 = 1+3j, \lambda_2 = 1-3j$$

.

(iii)

$$P(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) = \lambda^2 - 3\lambda + 3$$

$$P(\lambda) = 0 \Rightarrow \lambda_1 = \lambda_2 = 2$$

.

Solution 2

(i)

$$|3+2j| = \sqrt{3^2+2^2} = \sqrt{13}, \phi_1 = \arctan\frac{2}{3} \approx 0.588$$

(ii)

$$|2-j| = \sqrt{2^2+1^2} = \sqrt{5}, \phi_2 = \arctan\frac{-1}{2} \approx -0.464$$

(iii)

$$\left|\frac{3+2j}{2-j}\right| = \frac{|3+2j|}{|2-j|} = \frac{\sqrt{65}}{5}, \phi_3 = \phi_1 - \phi_2 = \arctan\frac{-1}{2} \approx 1.052$$

General rule: for two complex numbers $z_1 = |z_1|e^{j\phi_1}$ and $z_2 = |z_2|e^{j\phi_2}$,

$$\frac{z_1}{z_2} = me^{j\phi}$$

with

$$m = \frac{|z_1|}{|z_2|}, \qquad \phi = \phi_1 - \phi_2$$

Solution 3

The current through the capacitor is $I = C\frac{dV_C}{dt}$. The voltage acorss the inductor is $V_L = L\frac{dI}{dt}$. The voltage across the resistor is $V_R = RI$.

Now take $x_1 = V_C, x_2 = \frac{dV_C}{dt}$. Notice that

$$V_R = RI = RC\frac{dV_C}{dt} = RCx_2, V_L = L\frac{dI}{dt} = LC\frac{d^2V_C}{dt^2} = LC\dot{x}_2$$

Applying KVL, $V_C + V_L + V_R = V_S \Rightarrow x_1 + RCx_2 + LC\dot{x}_2 = V_S$.
Hence we should have

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{1}{RC}x_1 - \frac{L}{R}x_2 + \frac{1}{RC}V_S$$
$$\Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{LC} \end{pmatrix} V_S$$

Solution 4

(i) Pick $x_1 = x, x_2 = \dot{x}$, then

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} u$$

(ii) Pick $x_1 = x, x_2 = \dot{x}, x_3 = \ddot{x}$, then

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

Solution 5

(i) Across the capacitor, $I_C = C\frac{dV}{dt}$. Across the inductor, $V = L\frac{dI_L}{dt}$. Applying KCL,

$$I_C + I_L + I = 0$$

Take derivative on both sides of the above equation and substitute I with $g(V)$,

$$\frac{dI_C}{dt} + \frac{dI_L}{dt} + \frac{dg}{dt} = 0$$

$$\Rightarrow \frac{Cd^2V}{dt^2} + \frac{V}{L} + \frac{dg}{dV}\frac{dV}{dt} = 0$$

Notice that chain rule has been applied here for deriving the last term on the left.

(ii) Pick $x_1 = V, x_2 = \frac{dV}{dt}$, then the ODE in (i) becomes:

$$C\dot{x}_2 + \frac{x_1}{L} + \frac{dg}{dx_1}x_2 = 0$$

Hence the state space model of the system is:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{x_1}{LC} - \frac{dg}{Cdx_1}x_2\end{aligned}$$

(iii)

$$g(V) = -V + \frac{1}{3}V^3 \Rightarrow \frac{dg}{dV} = -1 + V^2$$

Hence the state space model in (ii) becomes

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{x_1}{LC} - \frac{(-1+x_1^2)x_2}{C}\end{pmatrix} =: f(x_1, x_2)$$

At equilibrium we must have $\dot{x}_1 = \dot{x}_2 = 0$, which means $f(x_1, x_2) = 0$. Inspecting on the first element of f, it suggests $x_2 = 0$. Replace x_2 with 0 in the second element of f and we see that it is 0 only if x_1 is 0. Therefore the only equilibrium of the system is the origin.

The system is linearized by taking the Jacobian of f at the origin:

$$\nabla f(0) = \begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{pmatrix}\Bigg|_0 = \begin{pmatrix} 0 & 1\\ -\frac{1}{LC} - \frac{2x_1x_2}{C} & \frac{1-x_1^2}{C}\end{pmatrix}\Bigg|_0 = \begin{pmatrix}0 & 1\\ -\frac{1}{LC} & \frac{1}{C}\end{pmatrix}$$

Hence the linearized system is:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{1}{LC} & \frac{1}{C}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$